Tangents and Normals

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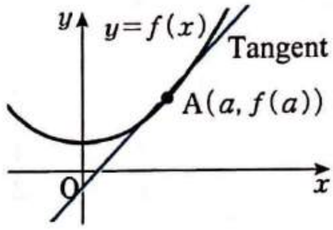
Given a curve $y=f(x)$, the slope of the tangent at point $\mathrm{A}(a, f(a))$ is $f'(a)$.
Therefore, the equation of the tangent is expressed as follows.



Equation of a Tangent

The equation of the tangent to the curve $y=f(x)$ at point $(a, f(a))$ is

$$y-f(a)=f'(a)(x-a)$$

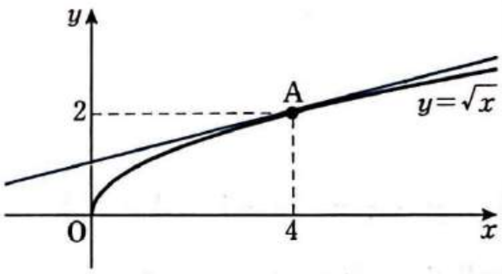
Find the equation of the tangent to each given curve at point A.

Ex. $y=\sqrt{x}$, $\mathrm{A}(4, 2)$

[Sol] Let $f(x)=\sqrt{x}$.

$$f'(x)=\frac{1}{2\sqrt{x}}$$

$$f'(4)=\frac{1}{4}$$



Therefore, the equation of the tangent is $y-2=\frac{1}{4}(x-4)$.

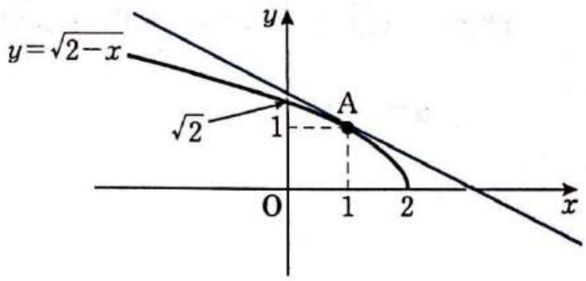
$$\therefore\ y=\frac{1}{4}x+1$$

(1) $y=\sqrt{2-x}$, $\mathrm{A}(1, 1)$

[Sol] Let $f(x)=\sqrt{2-x}$.

$$f'(x)=-\frac{1}{2\sqrt{2-x}}$$

$$f'(1)=-\frac{1}{2}$$



Therefore, the equation of the tangent is $y-1=-\frac{1}{2}(x-1)$.

$$\therefore\ y=-\frac{1}{2}x+\frac{3}{2}$$

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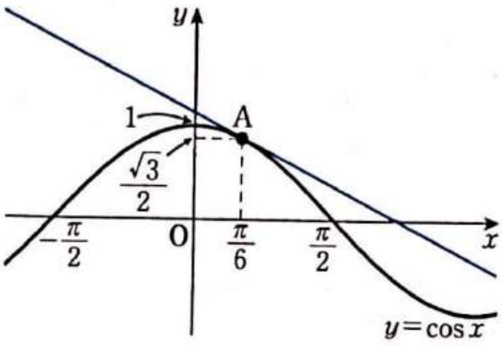
01b

(2) $y=\cos x$, $A\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$



[Sol] Let $f(x)=\cos x$.

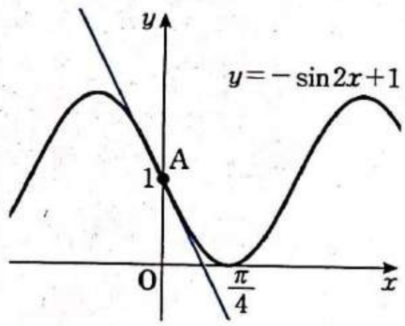
$$f'(x)=-\sin x$$

$$f'\left(\frac{\pi}{6}\right)=-\frac{1}{2}$$

Therefore, the equation of the tangent is $y-\frac{\sqrt{3}}{2}=-\frac{1}{2}\left(x-\frac{\pi}{6}\right)$.

$$\therefore\ \boldsymbol{y=-\frac{1}{2}x+\frac{\pi}{12}+\frac{\sqrt{3}}{2}}$$

(3) $y=-\sin 2x+1$, $A(0, 1)$



[Sol] Let $f(x)=-\sin 2x+1$.

$$f'(x)=-2\cos 2x$$

$$f'(0)=-2$$

Therefore, the equation of the tangent is $y-1=-2(x-0)$.

$$\therefore\ \boldsymbol{y=-2x+1}$$

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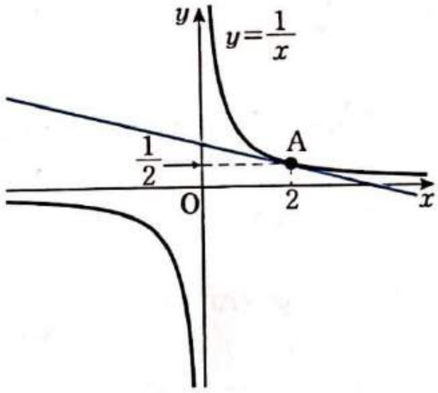
Find the equation of the tangent to each given curve at point A.

(1) $y=\dfrac{1}{x}$, $A\left(2, \dfrac{1}{2}\right)$



[Sol] Let $f(x)=\dfrac{1}{x}$.

$$f'(x)=-\frac{1}{x^2}$$

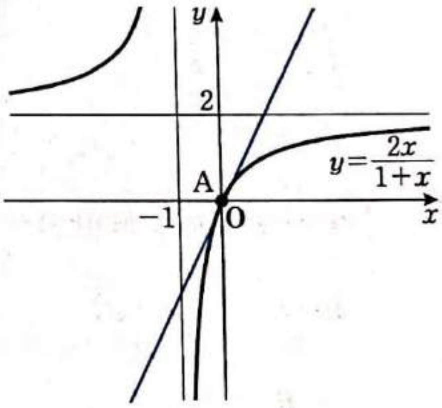
$$f'(2)=-\frac{1}{4}$$

Therefore, the equation of the tangent is

$$y-\frac{1}{2}=-\frac{1}{4}(x-2)$$

$$\therefore\ \boldsymbol{y=-\frac{1}{4}x+1}$$

(2) $y=\dfrac{2x}{1+x}$, $A(0, 0)$



[Sol] Let $f(x)=\dfrac{2x}{1+x}$.

$$f'(x)=\frac{2(1+x)-2x\cdot 1}{(1+x)^2}=\frac{2}{(1+x)^2}$$

$$f'(0)=2$$

Therefore, the equation of the tangent is

$$y-0=2(x-0)$$

$$\therefore\ \boldsymbol{y=2x}$$

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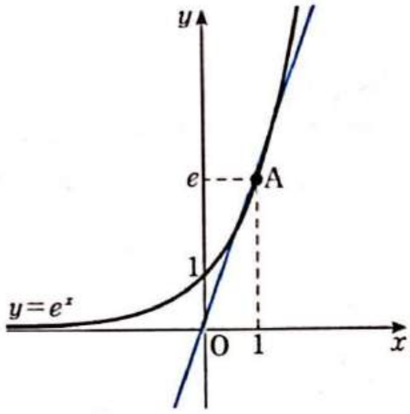
O2b

(3) $y=e^x$, $A(1, e)$



[Sol] Let $f(x)=e^x$.

$$f'(x)=e^x$$

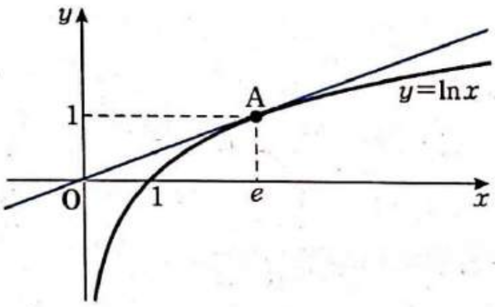
$$f'(1)=e$$

Therefore, the equation of the tangent is

$$y-e=e(x-1)$$

$$\therefore\ \boldsymbol{y=ex}$$

(4) $y=\ln x$, $A(e, 1)$



[Sol] Let $f(x)=\ln x$.

$$f'(x)=\frac{1}{x}$$

$$f'(e)=\frac{1}{e}$$

Therefore, the equation of the tangent is

$$y-1=\frac{1}{e}(x-e)$$

$$\therefore\ \boldsymbol{y=\frac{1}{e}x}$$

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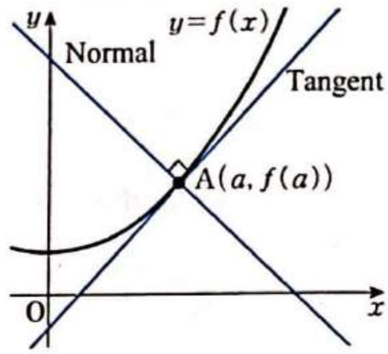
When a line which passes through point A on the curve is perpendicular to the tangent at point A, this line is called the ***normal*** to the curve at point A. The slope of the tangent at point $A(a, f(a))$ is $f'(a)$. Since the normal is perpendicular to the tangent, when $f'(a)\neq 0$, the slope of the normal is $-\dfrac{1}{f'(a)}$. ← Perpendicular Condition (M I 5)



Therefore, the equation of the normal is expressed as follows.

Equation of a Normal

The equation of the normal to the curve $y=f(x)$ at point $(a, f(a))$ is,

$$y-f(a)=-\frac{1}{f'(a)}(x-a) \text{ when } f'(a)\neq 0$$

$$x=a \text{ when } f'(a)=0$$

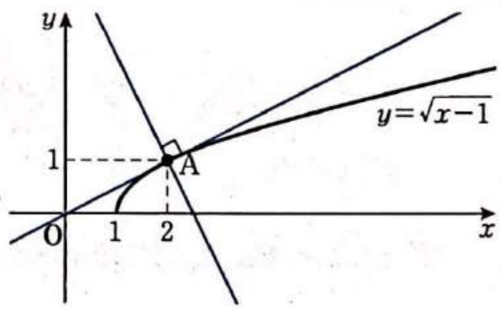
Find the equations of the tangent and the normal to each given curve at point A.

Ex. $y=\sqrt{x-1}$, $A(2, 1)$

[Sol] Let $f(x)=\sqrt{x-1}$.

$$f'(x)=\frac{1}{2\sqrt{x-1}}$$

$$f'(2)=\frac{1}{2}$$



Therefore, the equation of the tangent is $y-1=\frac{1}{2}(x-2)$.

$\therefore\ y=\frac{1}{2}x$

The equation of the normal is $y-1=\boxed{-2}(x-2)$. ←

$\therefore\ y=\boxed{-2x+5}$

The slope of the normal at point $(a, f(a))$ is $-\dfrac{1}{f'(a)}$.

Answers: -2, $-2x+5$

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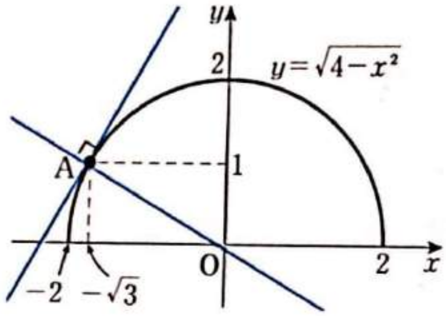
O3b

(1) $y=\sqrt{4-x^2}$, $A(-\sqrt{3}, 1)$



[Sol] Let $f(x)=\sqrt{4-x^2}$.

$$f'(x)=-\frac{x}{\sqrt{4-x^2}}$$

$$f'(-\sqrt{3})=\sqrt{3}$$

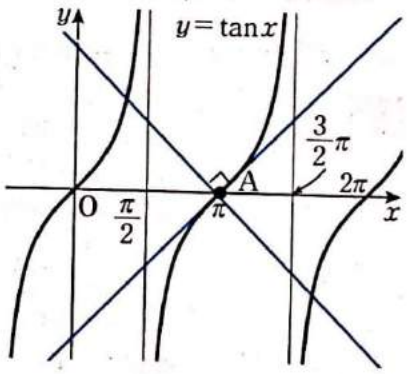
Therefore, **the equation of the tangent** is $y-1=\sqrt{3}[x-(-\sqrt{3})]$.

$\therefore\ \boldsymbol{y=\sqrt{3}x+4}$

The equation of the normal is $y-1=-\frac{\sqrt{3}}{3}[x-(-\sqrt{3})]$.

$\therefore\ \boldsymbol{y=-\frac{\sqrt{3}}{3}x}$

(2) $y=\tan x$, $A(\pi, 0)$



[Sol] Let $f(x)=\tan x$.

$$f'(x)=\frac{1}{\cos^2 x} \quad \Leftarrow \quad \boxed{(\tan x)'=\frac{1}{\cos^2 x}}$$

$$f'(\pi)=1$$

Therefore, **the equation of the tangent** is

$$y-0=1\cdot(x-\pi)$$

$\therefore\ \boldsymbol{y=x-\pi}$

The equation of the normal is $y-0=-1\cdot(x-\pi)$.

$\therefore\ \boldsymbol{y=-x+\pi}$

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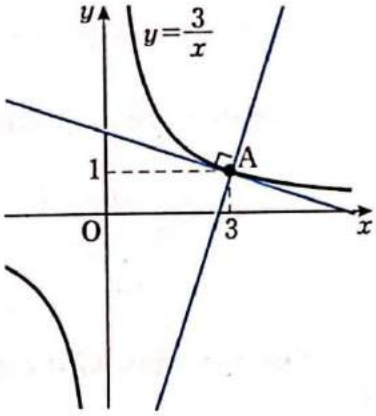
Find the equations of the tangent and the normal to each given curve at point A.

(1) $y=\dfrac{3}{x}$, A(3, 1)



[Sol] Let $f(x)=\dfrac{3}{x}$.

$$f'(x)=-\frac{3}{x^2}$$

$$f'(3)=-\frac{1}{3}$$

Therefore, **the equation of the tangent** is

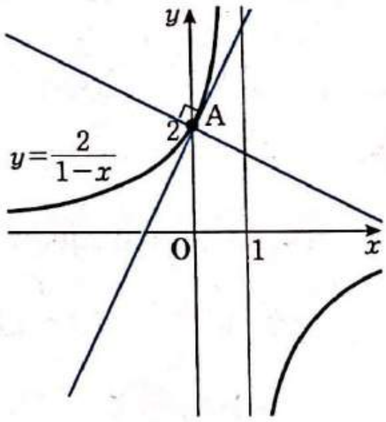
$$y-1=-\frac{1}{3}(x-3)$$

$$\therefore\ \boldsymbol{y=-\frac{1}{3}x+2}$$

The equation of the normal is $y-1=3(x-3)$.

$$\therefore\ \boldsymbol{y=3x-8}$$

(2) $y=\dfrac{2}{1-x}$, A(0, 2)



[Sol] Let $f(x)=\dfrac{2}{1-x}$.

$$f'(x)=-\frac{2\cdot(-1)}{(1-x)^2}=\frac{2}{(1-x)^2}$$

$$f'(0)=2$$

Therefore, **the equation of the tangent** is

$$y-2=2(x-0)$$

$$\therefore\ \boldsymbol{y=2x+2}$$

The equation of the normal is $y-2=-\dfrac{1}{2}(x-0)$.

$$\therefore\ \boldsymbol{y=-\frac{1}{2}x+2}$$

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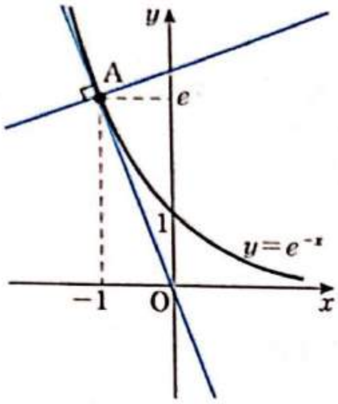
O4b

(3) $y=e^{-x}$, $A(-1, e)$



[Sol] Let $f(x)=e^{-x}$.

$$f'(x)=-e^{-x}$$

$$f'(-1)=-e$$

Therefore, the equation of the tangent is

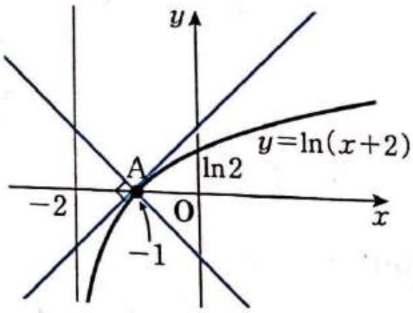
$$y-e=-e[x-(-1)]$$

$$\therefore\ y=-ex$$

The equation of the normal is $y-e=\frac{1}{e}[x-(-1)]$.

$$\therefore\ y=\frac{1}{e}x+\frac{1}{e}+e$$

(4) $y=\ln(x+2)$, $A(-1, 0)$



[Sol] Let $f(x)=\ln(x+2)$.

$$f'(x)=\frac{1}{x+2}$$

$$f'(-1)=1$$

Therefore, the equation of the tangent is

$$y-0=1\cdot[x-(-1)]$$

$$\therefore\ y=x+1$$

The equation of the normal is $y-0=-1\cdot[x-(-1)]$.

$$\therefore\ y=-x-1$$

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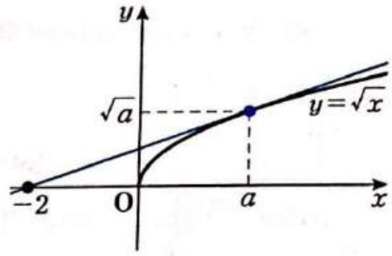
Ex. Given that a line passing through point $(-2, 0)$ is tangent to curve $y=\sqrt{x}$, find the equation of the tangent and the coordinates of the tangent point.



[Sol] Let $f(x)=\sqrt{x}$.

$$f'(x)=\frac{1}{2\sqrt{x}}$$

Let the coordinates of the tangent point be $(a, \sqrt{a})$.

The equation of the tangent is

$$y-\sqrt{a}=\frac{1}{2\sqrt{a}}(x-a)$$

Since this line passes through $(-2, 0)$,

$$0-\sqrt{a}=\frac{1}{2\sqrt{a}}(-2-a)$$

$$\therefore\ a=2$$

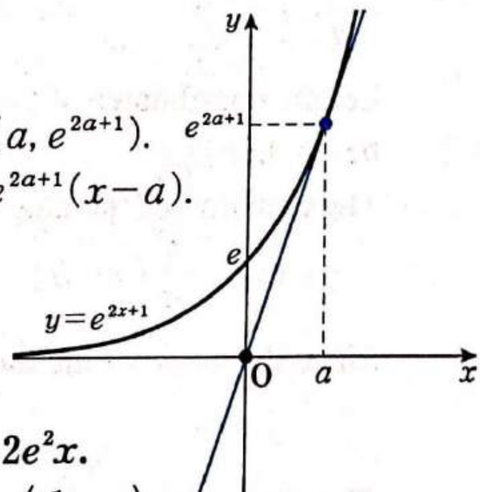
Substituting $a=2$ into $y-\sqrt{a}=\frac{1}{2\sqrt{a}}(x-a)$

Therefore, the equation of the tangent is $y=\frac{\sqrt{2}}{4}x+\frac{\sqrt{2}}{2}$.

Also, the coordinates of the tangent point are $(2, \sqrt{2})$.

Substituting $a=2$ into $(a, \sqrt{a})$

1. Given that a line passing through the origin is tangent to curve $y=e^{2x+1}$, find the equation of the tangent and the coordinates of the tangent point.



[Sol] Let $f(x)=e^{2x+1}$.

$$f'(x)=2e^{2x+1}$$

Let the coordinates of the tangent point be (a, e^{2a+1}).

The equation of the tangent is $y-e^{2a+1}=2e^{2a+1}(x-a)$.

Since this line passes through the origin,

$$0-e^{2a+1}=2e^{2a+1}(0-a)$$

Since $e^{2a+1}\neq 0$, $a=\frac{1}{2}$

Therefore, the equation of the tangent is $\boldsymbol{y=2e^2x}$.

Also, the coordinates of the tangent point are $\boldsymbol{\left(\frac{1}{2}, e^2\right)}$.

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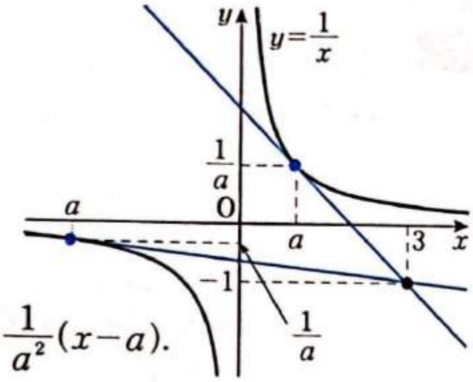
O5b

2. Given that a line passing through point $(3, -1)$ is tangent to curve $y=\frac{1}{x}$, find the equation of the tangent and the coordinates of the tangent point.



[Sol] Let $f(x)=\frac{1}{x}$.

$f'(x)=-\frac{1}{x^2}$

Let the coordinates of the tangent point be $\left(a, \frac{1}{a}\right)$.

The equation of the tangent is $y-\frac{1}{a}=-\frac{1}{a^2}(x-a)$.

Since this line passes through $(3, -1)$,

$-1-\frac{1}{a}=-\frac{1}{a^2}(3-a)$

$a^2+2a-3=0$

$(a+3)(a-1)=0$

$\therefore\ a=-3,\ 1$

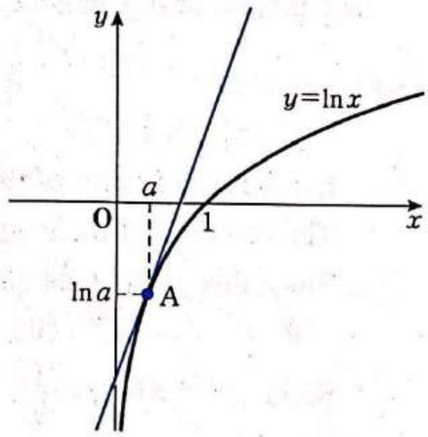
When $a=-3$, the equation of the tangent is $\boldsymbol{y=-\frac{1}{9}x-\frac{2}{3}}$.

Also, the coordinates of the tangent point are $\left(-3, -\frac{1}{3}\right)$.

When $a=1$, the equation of the tangent is $\boldsymbol{y=-x+2}$.

Also, the coordinates of the tangent point are $(1, 1)$.

3. Given that the slope of the tangent to curve $y=\ln x$ at point A is e, find the equation of the tangent at point A and the coordinates of point A.



[Sol] Let $f(x)=\ln x$.

$f'(x)=\frac{1}{x}$

Let the coordinates of point A be $(a, \ln a)$.

The equation of the tangent is

$y-\ln a=\frac{1}{a}(x-a)$

Since the slope of the tangent is e, $\frac{1}{a}=e$

$\therefore\ a=\frac{1}{e}$

Therefore, the equation of the tangent is $\boldsymbol{y=ex-2}$. $\boxed{\ln\frac{1}{e}=-\ln e=-1}$

Also, the coordinates of point A are $\left(\frac{1}{e}, -1\right)$.

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Ex. Given that two curves $y=ax^2$ and $y=\ln x$ have a common point P and the tangents to those two curves at point P overlap, find the value of constant a.

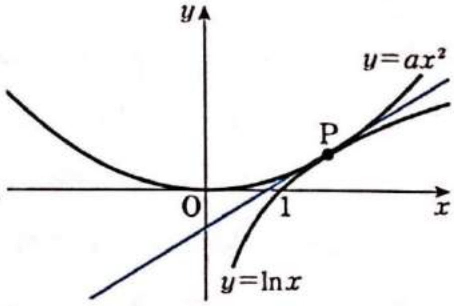
[Sol] Let $f(x)=ax^2$ and $g(x)=\ln x$.

Let the common point be $P(p, q)$.

$q=f(p)$, $q=g(p)$; therefore,

$ap^2=\ln p$ …①

Also, $f'(x)=2ax$, $g'(x)=\dfrac{1}{x}$



Since the slopes of the two tangents at point P are equal,

$2ap=\dfrac{1}{p}$, i.e. $ap^2=\dfrac{1}{2}$ …②

From ① and ②, $\ln p=\dfrac{1}{2}$; therefore, $p=\sqrt{e}$ ⟵ When $\ln M=x$, $M=e^x$

$\therefore\ a=\dfrac{1}{2e}$ ⟵ Substituting $p=\sqrt{e}$ into ① or ②

1. Given that two curves $y=e^x$ and $y=a\sqrt{x}$ have a common point P and the tangents to those two curves at point P overlap, find the value of constant a. $(a\neq 0)$

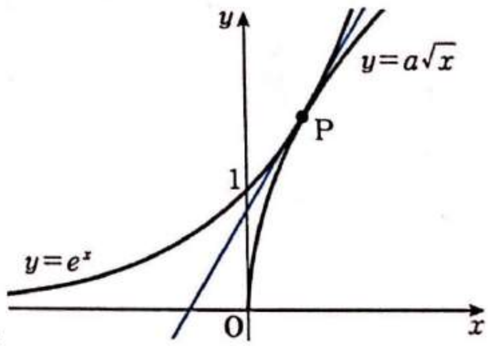
[Sol] Let $f(x)=e^x$ and $g(x)=a\sqrt{x}$.

Let the common point be $P(p, q)$.

$q=f(p)$, $q=g(p)$; therefore,

$e^p=a\sqrt{p}$ …①

Also, $f'(x)=e^x$, $g'(x)=\dfrac{a}{2\sqrt{x}}$



Since the slopes of the two tangents at point P are equal,

$e^p=\dfrac{a}{2\sqrt{p}}$ …②

From ① and ②, $a\sqrt{p}=\dfrac{a}{2\sqrt{p}}$. Since $a\neq 0$, $p=\dfrac{1}{2}$.

$\therefore\ a=\sqrt{2e}$

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O6b

2. Given that two curves $y=2\sin x$ and $y=a-\cos 2x$ have a common point P in the interval $0 < x < \frac{\pi}{2}$ and the tangents to those two curves at point P overlap, find the value of constant a.

[Sol] Let $f(x)=2\sin x$ and $g(x)=a-\cos 2x$.

Let the common point be $\mathrm{P}(p, q)$. $\left(0 < p < \frac{\pi}{2}\right)$

$q=f(p)$, $q=g(p)$; therefore,

$2\sin p=a-\cos 2p \quad \cdots①$

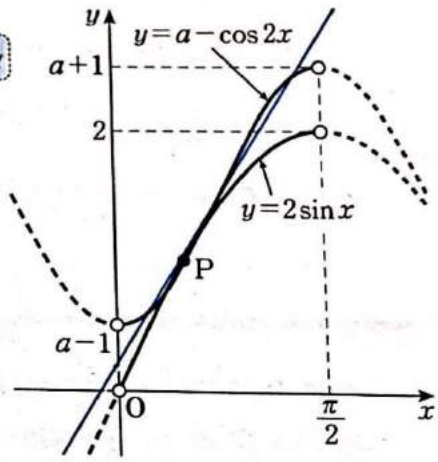
Also, $f'(x)=2\cos x$, $g'(x)=2\sin 2x$

Since the slopes of the two tangents at point P are equal,

$2\cos p=2\sin 2p$

$\cos p=2\sin p\cos p$ $\quad \boxed{\sin 2\alpha=2\sin\alpha\cos\alpha}$

$\cos p(2\sin p-1)=0$



Since $0 < p < \frac{\pi}{2}$, $\cos p \neq 0$

$\therefore\ \sin p=\frac{1}{2}$

$\therefore\ p=\frac{\pi}{6} \quad \cdots②$

From ① and ②, $2\sin\frac{\pi}{6}=a-\cos\frac{\pi}{3}$

$\therefore\ a=\frac{3}{2}$

Note Summary

When two curves $y=f(x)$ and $y=g(x)$ have a common point (where p is the x-coordinate) and the tangents to those two curves at the point overlap, then

$f(p)=g(p)$ and also $f'(p)=g'(p)$

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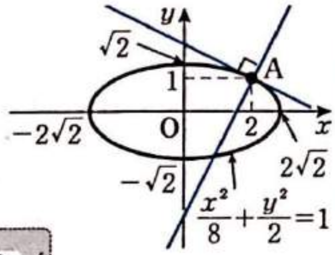
1. Find the equations of the tangent and the normal to each given curve at point A.

Ex. $\dfrac{x^2}{8}+\dfrac{y^2}{2}=1$, A$(2, 1)$



[Sol] Differentiating both sides of $\dfrac{x^2}{8}+\dfrac{y^2}{2}=1$ with respect to x,

$$\frac{2x}{8}+\frac{2y}{2}\cdot y'=0 \quad \longleftarrow \quad \frac{d}{dx}y^2=\frac{d}{dy}y^2\cdot\frac{dy}{dx}=2yy'$$

Therefore, when $y\neq 0$, $y'=-\dfrac{x}{4y}$

※

Thus, the slope of the tangent at point A is $-\dfrac{2}{4\cdot 1}=-\dfrac{1}{2}$.

Therefore, the equation of the tangent is $y-1=-\dfrac{1}{2}(x-2)$.

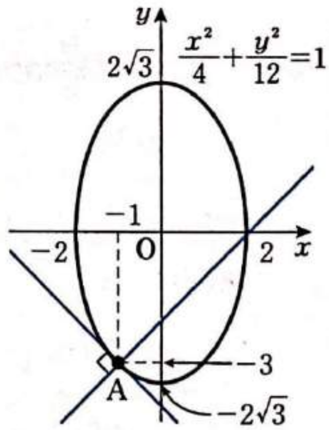
$\therefore\ y=-\dfrac{1}{2}x+2$

※ Substituting $x=2$ and $y=1$

The equation of the normal is $y-1=2(x-2)$.

$\therefore\ y=2x-3$

(1) $\dfrac{x^2}{4}+\dfrac{y^2}{12}=1$, A$(-1, -3)$



[Sol] Differentiating both sides of $\dfrac{x^2}{4}+\dfrac{y^2}{12}=1$ with respect to x,

$$\frac{2x}{4}+\frac{2y}{12}\cdot y'=0$$

Therefore, when $y\neq 0$, $y'=-\dfrac{3x}{y}$

Thus, the slope of the tangent at point A is $-\dfrac{3\cdot(-1)}{-3}=-1$.

Therefore, **the equation of the tangent** is $y-(-3)=-1\cdot[x-(-1)]$.

$\therefore\ \boldsymbol{y=-x-4}$

The equation of the normal is $y-(-3)=1\cdot[x-(-1)]$.

$\therefore\ \boldsymbol{y=x-2}$

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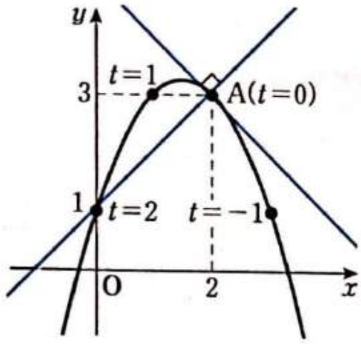
O7b

2. For each given curve represented by the equations using the parameter, find the equations of the tangent and the normal to the curve at point A indicated in the parentheses ( ).

(1) $\begin{cases} x=2-t \\ y=3+t-t^2 \end{cases}$ $(t=0)$



[Sol] $\dfrac{dx}{dt}=\boxed{-1}$, $\dfrac{dy}{dt}=\boxed{1-2t}$

$\therefore\ \dfrac{dy}{dx}=\dfrac{1-2t}{-1}=2t-1$ ⟵ $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Therefore, the slope of the tangent at point A is $2\cdot 0-1=-1$.

The coordinates of point A are $(2, 3)$. ⟵ Substituting $t=0$ into $x=2-t$ and $y=3+t-t^2$

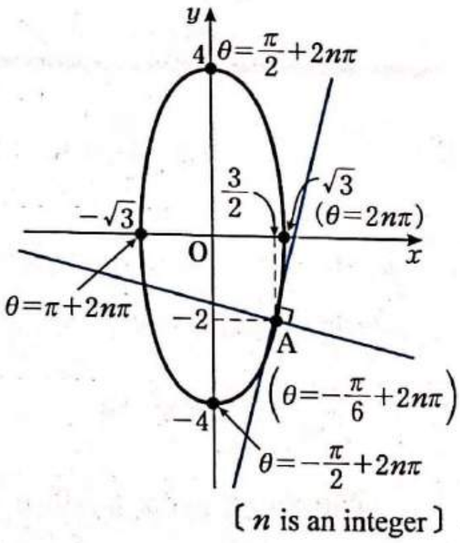
Thus, **the equation of the tangent** is $y-3=-1\cdot(x-2)$.

$\therefore\ \boldsymbol{y=-x+5}$

The equation of the normal is $y-3=1\cdot(x-2)$.

$\therefore\ \boldsymbol{y=x+1}$

(2) $\begin{cases} x=\sqrt{3}\cos\theta \\ y=4\sin\theta \end{cases}$ $\left(\theta=-\dfrac{\pi}{6}\right)$



[Sol] $\dfrac{dx}{d\theta}=-\sqrt{3}\sin\theta$, $\dfrac{dy}{d\theta}=4\cos\theta$

$\therefore\ \dfrac{dy}{dx}=\dfrac{4\cos\theta}{-\sqrt{3}\sin\theta}=-\dfrac{4\sqrt{3}\cos\theta}{3\sin\theta}$

Therefore, the slope of the tangent at point A is

$$-\frac{4\sqrt{3}\cos\left(-\frac{\pi}{6}\right)}{3\sin\left(-\frac{\pi}{6}\right)}=4$$

The coordinates of point A are $\left(\dfrac{3}{2}, -2\right)$.

Thus, **the equation of the tangent** is $y-(-2)=4\left(x-\dfrac{3}{2}\right)$.

$\therefore\ \boldsymbol{y=4x-8}$

The equation of the normal is $y-(-2)=-\dfrac{1}{4}\left(x-\dfrac{3}{2}\right)$.

$\therefore\ \boldsymbol{y=-\dfrac{1}{4}x-\dfrac{13}{8}}$

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Tangents and Normals

Name

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Ex.

Prove that the equation of the tangent to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at point $A(x_1, y_1)$ is $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$.

[Sol] Differentiating both sides of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to x,

$$\frac{2x}{a^2}+\frac{2y}{b^2}\cdot y'=0 \quad \longleftarrow \quad \frac{d}{dx}y^2=\frac{d}{dy}y^2\cdot\frac{dy}{dx}=2yy'$$

Therefore, when $y\neq 0$, $y'=\boxed{-\frac{b^2x}{a^2y}}$

When $y_1\neq 0$, the equation of the tangent at point A is

$$y-y_1=\boxed{-\frac{b^2x_1}{a^2y_1}}(x-x_1)$$

Multiplying both sides by $\frac{y_1}{b^2}$ to be rearranged

$$\therefore \frac{x_1x}{a^2}+\frac{y_1y}{b^2}=\boxed{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}} \quad \cdots ①$$

Since point A lies on ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \quad \cdots ②$

From ① and ②, $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1 \quad \cdots ③$

Point A lies on ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

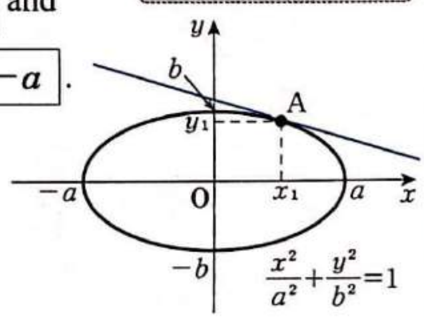
Also, when $y_1=0$, $x_1=\pm a$; therefore,

the tangent at point $(a, 0)$ is $x=\boxed{a}$ and

the tangent at point $(-a, 0)$ is $x=\boxed{-a}$.

This satisfies ③.

$$\therefore \frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$$



[When $a>0$ and $b>0$]

Answers: $-\frac{b^2x}{a^2y}$, $-\frac{b^2x_1}{a^2y_1}$, $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$, a, $-a$

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O8b

1. Prove that the equation of the tangent to parabola $y^2=4px$ at point $A(x_1, y_1)$ is $y_1y=2p(x+x_1)$.

[Sol] Differentiating both sides of $y^2=4px$ with respect to x,

$$2yy'=4p$$

Therefore, when $y\neq0$, $y'=\frac{2p}{y}$

When $y_1\neq0$, the equation of the tangent at point A is

$$y-y_1=\frac{2p}{y_1}(x-x_1)$$

$$\therefore\ y_1y=2p(x-x_1)+y_1^2 \quad \cdots①$$

Since point A lies on parabola $y^2=4px$, $y_1^2=4px_1$ $\cdots②$

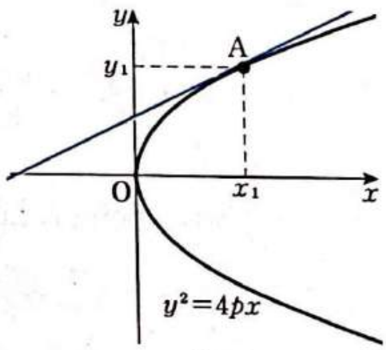
From ① and ②, $y_1y=2p(x-x_1)+4px_1=2p(x+x_1)$ $\cdots③$

Also, when $y_1=0$, $x_1=0$; therefore,

the tangent at point $(0, 0)$ is $x=0$.

This satisfies ③.

$$\therefore\ y_1y=2p(x+x_1)$$



〔When $p>0$〕

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Tangents and Normals

Name

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1. Given that graphs of $y=x^2+ax+b$ and $y=\frac{8}{x}$ intersect at point $(2, 4)$ and the tangents at this point intersect perpendicularly, find the values of a and b.

[Sol] Let $f(x)=x^2+ax+b$ and $g(x)=\frac{8}{x}$.

Since $y=f(x)$ passes through point $(2, 4)$,

$4=4+2a+b$, i.e. $2a+b=0$ ⋯①

Also, $f'(x)=2x+a,\ g'(x)=-\frac{8}{x^2}$

The slope of each tangent at point $(2, 4)$ is

$f'(2)=a+4,\ g'(2)=-2$

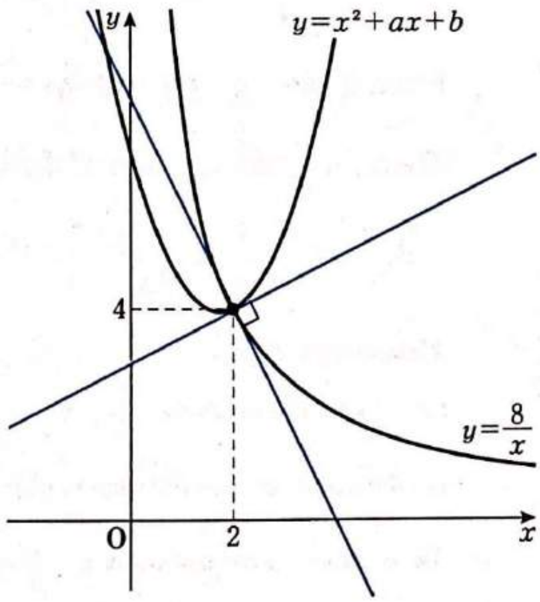
Since the tangents at point $(2, 4)$ intersect perpendicularly,

$(a+4)\cdot(-2)=-1$ ⟵ Perpendicular Condition (M15)

$\therefore\ a=-\frac{7}{2}$ ⋯②

From ① and ②, $b=7$

$\therefore\ \boldsymbol{a=-\frac{7}{2},\ b=7}$



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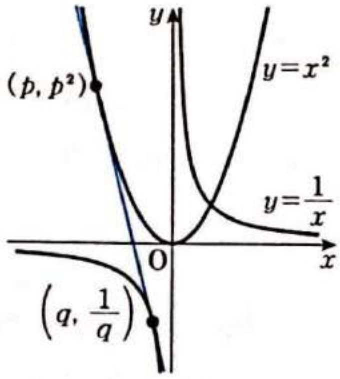
O9b

2. Find the equation of the line tangent to both curves $y=x^2$ and $y=\frac{1}{x}$.

[Sol] Let $f(x)=x^2$ and $g(x)=\frac{1}{x}$.



$f'(x)=2x$, $g'(x)=-\frac{1}{x^2}$

Let the coordinates of the tangent points of the line and two curves $y=f(x)$ and $y=g(x)$ be (p, p^2) and $\left(q, \frac{1}{q}\right)$ respectively.

The equations of the tangents are

$$y-p^2=2p(x-p), \text{ i.e. } y=2px-p^2 \quad \cdots①$$

$$y-\frac{1}{q}=-\frac{1}{q^2}(x-q), \text{ i.e. } y=-\frac{1}{q^2}x+\frac{2}{q} \quad \cdots②$$

Since two lines ① and ② overlap,

When two lines $y=m_1x+n_1$ and $y=m_2x+n_2$ overlap, $m_1=m_2$ and $n_1=n_2$.

$$2p=-\frac{1}{q^2} \quad \cdots③$$

$$-p^2=\frac{2}{q} \quad \cdots④$$

From ③ and ④, $p=-2$, $q=-\frac{1}{2}$

From ③, $p=-\frac{1}{2q^2}$
Substituting this into ④,
$-\left(-\frac{1}{2q^2}\right)^2=\frac{2}{q}$
$\therefore\ 8q^3=-1$

Therefore, the equation of the line is $\boldsymbol{y=-4x-4}$.

Substituting $p=-2$ into ① or $q=-\frac{1}{2}$ into ②

Alternative Solution

Let $f(x)=x^2$ and $g(x)=\frac{1}{x}$. $g'(x)=-\frac{1}{x^2}$

Let the coordinates of the tangent point of the line and the curve $y=g(x)$ be $\left(a, \frac{1}{a}\right)$.

The equation of the tangent is $y-\frac{1}{a}=-\frac{1}{a^2}(x-a)$.

Since this line is tangent to the curve $y=f(x)$, $x^2=-\frac{1}{a^2}(x-a)+\frac{1}{a}$

So, let the discriminant of $a^2x^2+x-2a=0$ be D.

$$D=1^2-4a^2\cdot(-2a)=8a^3+1=(2a+1)(4a^2-2a+1)=0$$

Discriminant (K76)

$$\therefore\ a=-\frac{1}{2}$$

$4a^2-2a+1$
$=4\left(a-\frac{1}{4}\right)^2+\frac{3}{4}>0$

Therefore, the equation of the line is $\boldsymbol{y=-4x-4}$.

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Tangents and Normals

Name

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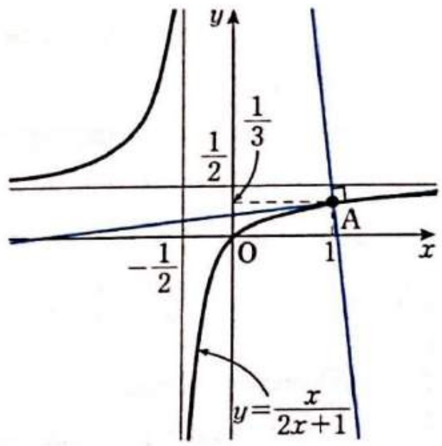
1. Find the equations of the tangent and the normal to each given curve at point A.

➡ O3

(1) $y=\dfrac{x}{2x+1}$, $\mathrm{A}\left(1, \dfrac{1}{3}\right)$



[Sol] Let $f(x)=\dfrac{x}{2x+1}$.

$$f'(x)=\frac{1\cdot(2x+1)-x\cdot 2}{(2x+1)^2}=\frac{1}{(2x+1)^2}$$

$$f'(1)=\frac{1}{9}$$

Therefore, **the equation of the tangent** is

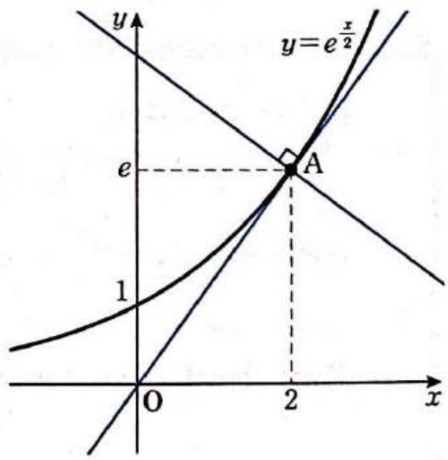
$$y-\frac{1}{3}=\frac{1}{9}(x-1)$$

$$\therefore\ \boldsymbol{y=\frac{1}{9}x+\frac{2}{9}}$$

The equation of the normal is $y-\dfrac{1}{3}=-9(x-1)$.

$$\therefore\ \boldsymbol{y=-9x+\frac{28}{3}}$$

(2) $y=e^{\frac{x}{2}}$, $\mathrm{A}(2, e)$



[Sol] Let $f(x)=e^{\frac{x}{2}}$.

$$f'(x)=\frac{1}{2}e^{\frac{x}{2}}$$

$$f'(2)=\frac{1}{2}e$$

Therefore, **the equation of the tangent** is

$$y-e=\frac{1}{2}e(x-2)$$

$$\therefore\ \boldsymbol{y=\frac{1}{2}ex}$$

The equation of the normal is $y-e=-\dfrac{2}{e}(x-2)$.

$$\therefore\ \boldsymbol{y=-\frac{2}{e}x+\frac{4}{e}+e}$$

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O10b

2. Given that a line passing through point $(0, -3)$ is tangent to curve $y=x\ln x$, find the equation of the tangent and the coordinates of the tangent point.

➡ O5

[Sol] Let $f(x)=x\ln x$.

$$f'(x)=1\cdot\ln x+x\cdot\frac{1}{x}=\ln x+1$$

Let the coordinates of the tangent point be $(a, a\ln a)$.

The equation of the tangent is

$y-a\ln a=(\ln a+1)(x-a)$

Since this line passes through $(0, -3)$,

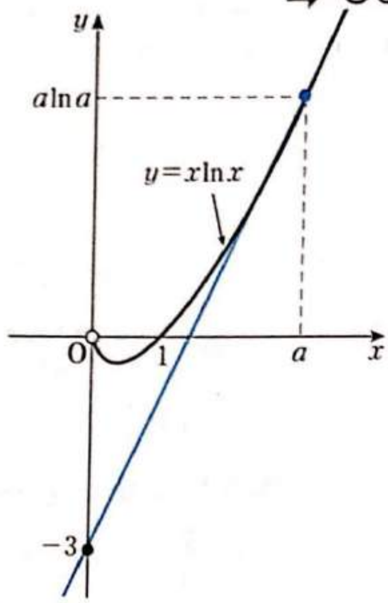
$-3-a\ln a=(\ln a+1)(0-a)$

$\therefore\ a=3$

Therefore, the equation of the tangent is

$\mathbf{y=(\ln 3+1)x-3}$

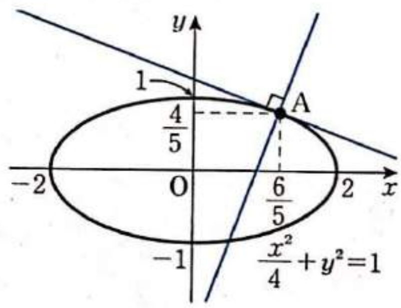
Also, the coordinates of the tangent point are $\mathbf{(3, 3\ln 3)}$.



3. Find the equations of the tangent and the normal to the following curve at point A.

➡ O7

$$\frac{x^2}{4}+y^2=1,\quad A\left(\frac{6}{5}, \frac{4}{5}\right)$$



[Sol] Differentiating both sides of $\frac{x^2}{4}+y^2=1$ with respect to x,

$$\frac{2x}{4}+2yy'=0$$

Therefore, when $y\neq 0$, $y'=-\frac{x}{4y}$

Thus, the slope of the tangent at point A is $-\frac{\frac{6}{5}}{4\cdot\frac{4}{5}}=-\frac{3}{8}$.

Therefore, **the equation of the tangent** is $y-\frac{4}{5}=-\frac{3}{8}\left(x-\frac{6}{5}\right)$.

$\therefore\ \mathbf{y=-\frac{3}{8}x+\frac{5}{4}}$

The equation of the normal is $y-\frac{4}{5}=\frac{8}{3}\left(x-\frac{6}{5}\right)$.

$\therefore\ \mathbf{y=\frac{8}{3}x-\frac{12}{5}}$

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Increasing/Decreasing Functions and Relative Extreme Values

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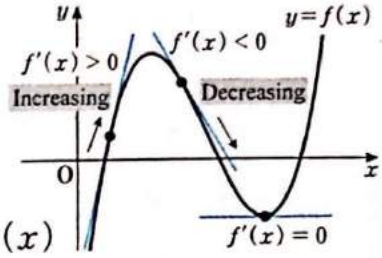


The slope of the tangent to a continuous function $y=f(x)$ at $x=a$ is $f'(a)$. From this,
if $f'(a)>0$, the tangent has a positive slope;
if $f'(a)<0$, the tangent has a negative slope; and
if $f'(a)=0$, the slope of the tangent is 0.
At the point close to $x=a$, the graph of function $y=f(x)$ is almost equal to the tangent. Therefore, the following statement is true.

Increasing/Decreasing Functions

Given that a function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b),
(i) if $f'(x)>0$ on the interval (a, b), $f(x)$ increases on the interval $[a, b]$;
(ii) if $f'(x)<0$ on the interval (a, b), $f(x)$ decreases on the interval $[a, b]$;
(iii) if $f'(x)=0$ on the interval (a, b), $f(x)$ is a constant on the interval $[a, b]$.

For each given function, determine where the function increases/decreases, and then trace the corresponding graph.

Ex. $y=x+2\sin x \quad (0 \leqslant x \leqslant \pi)$

[Sol] $y'=1+2\cos x$ — Considering $y'=0$ on the open interval

When $y'=0$ in $0<x<\pi$, $\cos x=-\dfrac{1}{2}$, i.e. $x=\dfrac{2}{3}\pi$

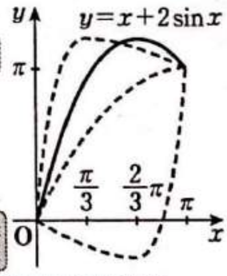
Creating the variation table,

x	0	$\cdots$	$\frac{2}{3}\pi$	$\cdots$	π
y'		$+$	0	$-$	
y	0	$\nearrow$	$\frac{2}{3}\pi+\sqrt{3}$	$\searrow$	π

Use $\cdots$ excluding the point where $y'=0$ and both ends.

When $y'>0$, use $+$ and when $y'<0$, use $-$. Nothing has to be written for either end of the domain.

$\nearrow$ means increasing, and $\searrow$ means decreasing.



Therefore,

it increases in $\boxed{0} \leqslant x \leqslant \boxed{\frac{2}{3}\pi}$, and decreases in $\boxed{\frac{2}{3}\pi} \leqslant x \leqslant \boxed{\pi}$.

The graph is as shown on the right.

The interval of increasing/decreasing includes both ends of the domain.

Answers: 0, $\frac{2}{3}\pi$, $\frac{2}{3}\pi$, π

【Reference】L61

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011b

(1) $y=x+2\cos x \quad \left(-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}\right)$

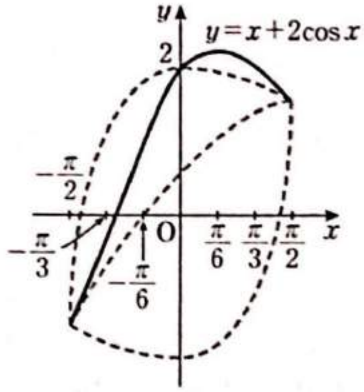
[Sol] $y'=1-2\sin x$

When $y'=0$ in $-\frac{\pi}{2}<x<\frac{\pi}{2}$,

$\sin x=\frac{1}{2}$, i.e. $x=\frac{\pi}{6}$

Creating the variation table,

x	$-\frac{\pi}{2}$	$\cdots$	$\frac{\pi}{6}$	$\cdots$	$\frac{\pi}{2}$
y'		$+$	0	$-$	
y	$-\frac{\pi}{2}$	$\nearrow$	$\frac{\pi}{6}+\sqrt{3}$	$\searrow$	$\frac{\pi}{2}$



Therefore, it **increases in** $-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{6}$, and **decreases in** $\frac{\pi}{6} \leqq x \leqq \frac{\pi}{2}$.

The graph is as shown on the right.

(2) $y=x\sin x+\cos x \quad (0 \leqq x \leqq 2\pi)$

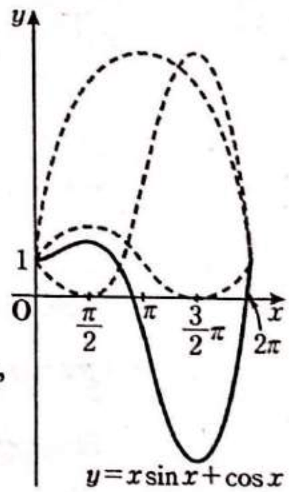
[Sol] $y'=(1\cdot\sin x+x\cos x)-\sin x=x\cos x$

When $y'=0$ in $0<x<2\pi$,

$\cos x=0$, i.e. $x=\frac{\pi}{2}, \frac{3}{2}\pi$

Creating the variation table,

x	0	$\cdots$	$\frac{\pi}{2}$	$\cdots$	$\frac{3}{2}\pi$	$\cdots$	2π
y'		$+$	0	$-$	0	$+$	
y	1	$\nearrow$	$\frac{\pi}{2}$	$\searrow$	$-\frac{3}{2}\pi$	$\nearrow$	1



Therefore, it **increases in** $0 \leqq x \leqq \frac{\pi}{2}$, $\frac{3}{2}\pi \leqq x \leqq 2\pi$,

and **decreases in** $\frac{\pi}{2} \leqq x \leqq \frac{3}{2}\pi$.

The graph is as shown on the right.

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Increasing/Decreasing Functions and Relative Extreme Values

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For each given function, determine where the function increases/decreases, and then trace the corresponding graph.

Ex. $y=x+\dfrac{1}{x}$

[Sol] The domain is $x\neq 0$. ← Checking the domain (denominator $\neq 0$)

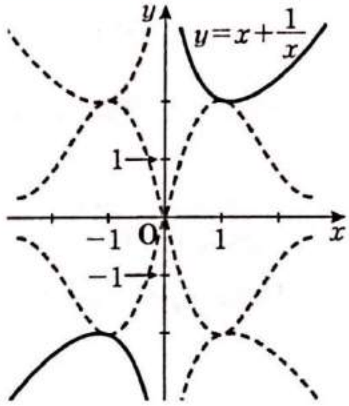
$$y'=1-\frac{1}{x^2}=\frac{(x+1)(x-1)}{x^2}$$

Draw a slash (/) for outside the domain.

When $y'=0$, $x=\pm 1$

Creating the variation table,

x	$\cdots$	-1	$\cdots$	0	$\cdots$	1	$\cdots$
y'	$+$	0	$-$	/	$-$	0	$+$
y	↗	-2	↘	/	↘	2	↗



Therefore, it increases in $x\leqq -1$, $1\leqq x$,
and decreases in $-1\leqq x<0$, $0<x\leqq 1$.
The graph is as shown on the right.

(1) $y=\dfrac{x^2+x+3}{x+3}$

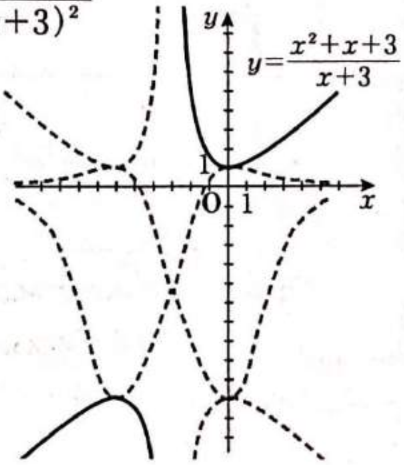
[Sol] The domain is $x\neq -3$.

$$y'=\frac{(2x+1)(x+3)-(x^2+x+3)\cdot 1}{(x+3)^2}=\frac{x(x+6)}{(x+3)^2}$$

When $y'=0$, $x=-6,\ 0$

Creating the variation table,

x	$\cdots$	-6	$\cdots$	-3	$\cdots$	0	$\cdots$
y'	$+$	0	$-$	/	$-$	0	$+$
y	↗	-11	↘	/	↘	1	↗



Therefore, it **increases in $x\leqq -6$, $0\leqq x$,**
and **decreases in $-6\leqq x<-3$, $-3<x\leqq 0$.**
The graph is as shown on the right.

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O12b

(2) $y=\sqrt{x}+\sqrt{4-x}$

[Sol] Since $x \geqq 0$, $4-x \geqq 0$, the domain is $0 \leqq x \leqq 4$. ⟵ The expression in each square root $\geqq 0$

$$y'=\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{4-x}}=\frac{\sqrt{4-x}-\sqrt{x}}{2\sqrt{x}\sqrt{4-x}}$$

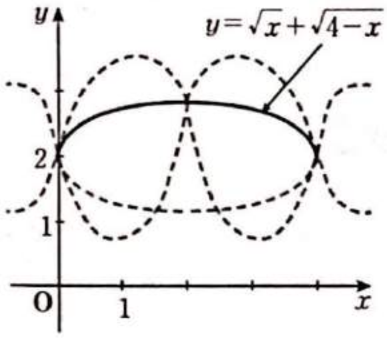
When $y'=0$ in $0<x<4$,

$\sqrt{4-x}=\sqrt{x}$, i.e. $x=2$ ⟵ Squaring both sides, $4-x=x$

Creating the variation table,

x	0	$\cdots$	2	$\cdots$	4
y'	/	+	0	−	/
y	2	↗	$2\sqrt{2}$	↘	2

⟵ Since $y'=\dfrac{\sqrt{4-x}-\sqrt{x}}{2\sqrt{x}\sqrt{4-x}}$, y' is not defined when $x=0$, 4. However, y is defined.



Therefore, it **increases in** $0 \leqq x \leqq 2$, and **decreases in** $2 \leqq x \leqq 4$.

The graph is as shown on the right.

(3) $y=\ln(x+1)-x$

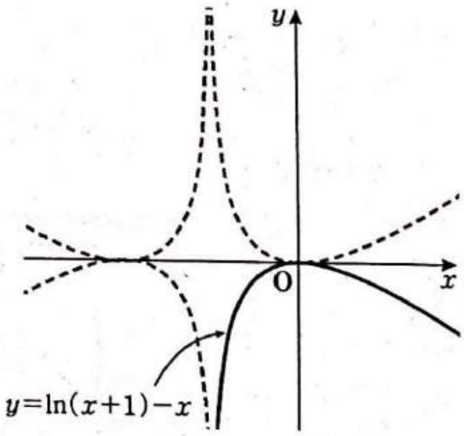
[Sol] Since $x+1>0$, the domain is $x>-1$. ⟵ Antilogarithm >0

$$y'=\frac{1}{x+1}-1=-\frac{x}{x+1}$$

When $y'=0$ in $x>-1$, $x=0$

Creating the variation table,

x	−1	$\cdots$	0	$\cdots$
y'	/	+	0	−
y	/	↗	0	↘



Therefore, it **increases in** $-1<x \leqq 0$,

and **decreases in** $0 \leqq x$.

The graph is as shown on the right.

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Increasing/Decreasing Functions and Relative Extreme Values

Name

Date / /

Time : to :

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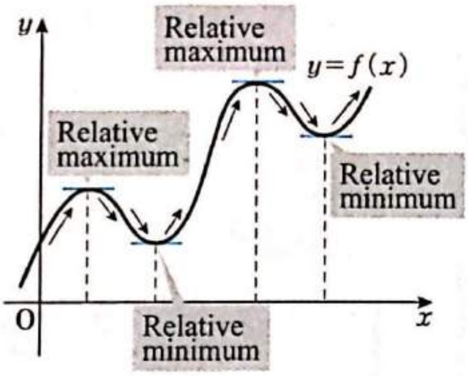
When a continuous function $f(x)$ switches from increasing to decreasing as x increases through a, the function $f(x)$ is said to have a ***relative maximum*** at $x=a$ and $f(a)$ is called the ***relative maximum value***. When the function $f(x)$ switches from decreasing to increasing as x increases through a, the function $f(x)$ is said to have a ***relative minimum*** at $x=a$ and $f(a)$ is called the ***relative minimum value***.



The relative maximum value and the relative minimum value are collectively called the ***relative extreme values***.

For each given function, find the relative extreme values, and then draw its graph.

Ex. $y=\sin^2 x+2\sin x \quad (0 \leqq x \leqq 2\pi)$

[Sol] $y'=2\sin x\cos x+2\cos x$

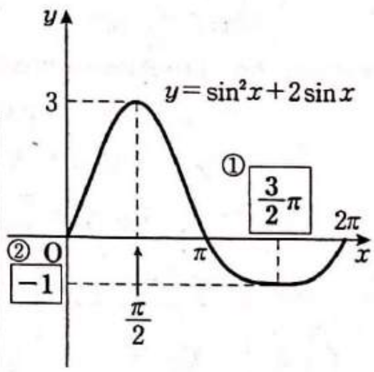
$=2\cos x(\sin x+1)$

When $y'=0$ in $0<x<2\pi$, $\cos x=0$, $\sin x=-1$,

i.e. $x=\frac{\pi}{2}$, $\boxed{\frac{3}{2}\pi}$

Creating the variation table,

x	0	$\cdots$	$\frac{\pi}{2}$	$\cdots$	$\frac{3}{2}\pi$	$\cdots$	2π
y'		$+$	0	$-$	0	$+$	
y	0	↗	3	↘	-1	↗	0



Therefore, the relative maximum value is $\boxed{3}$, at $x=\frac{\pi}{2}$ and

the relative minimum value is $\boxed{-1}$, at $x=\boxed{\frac{3}{2}\pi}$.

The graph is as shown on the right.

Answers: $\frac{3}{2}\pi$,

x	0	$\cdots$	$\frac{\pi}{2}$	$\cdots$	$\frac{3}{2}\pi$	$\cdots$	2π
y'		$+$	0	$-$	0	$+$	
y	0	↗	3	↘	-1	↗	0

, 3, −1, $\frac{3}{2}\pi$, (on the graph) ① $\frac{3}{2}\pi$, ② −1

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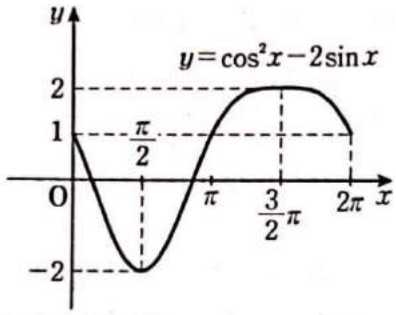
013b

(1) $y=\cos^2 x-2\sin x \quad (0 \leqq x \leqq 2\pi)$



[Sol] $y'=2\cos x\cdot(-\sin x)-2\cos x$

$=-2\cos x(\sin x+1)$

When $y'=0$ in $0<x<2\pi$,

$\cos x=0,\ \sin x=-1$, i.e. $x=\frac{\pi}{2},\ \frac{3}{2}\pi$

Creating the variation table,

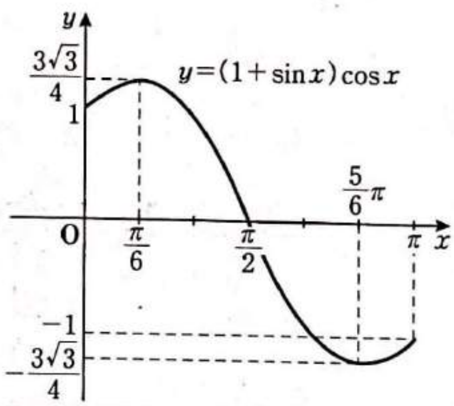
x	0	$\cdots$	$\frac{\pi}{2}$	$\cdots$	$\frac{3}{2}\pi$	$\cdots$	2π
y'		$-$	0	$+$	0	$-$	
y	1	↘	-2	↗	2	↘	1

Therefore, **the relative minimum value is -2, at $x=\frac{\pi}{2}$** and

the relative maximum value is 2, at $x=\frac{3}{2}\pi$.

The graph is as shown on the right.

(2) $y=(1+\sin x)\cos x \quad (0 \leqq x \leqq \pi)$



[Sol] $y'=\cos x\cdot\cos x+(1+\sin x)\cdot(-\sin x)$

$=\cos^2 x-\sin x-\sin^2 x$

$=-2\sin^2 x-\sin x+1$

$=-(\sin x+1)(2\sin x-1)$

When $y'=0$ in $0<x<\pi$,

$\sin x=\frac{1}{2}$, i.e. $x=\frac{\pi}{6},\ \frac{5}{6}\pi$

Creating the variation table,

x	0	$\cdots$	$\frac{\pi}{6}$	$\cdots$	$\frac{5}{6}\pi$	$\cdots$	π
y'		$+$	0	$-$	0	$+$	
y	1	↗	$\frac{3\sqrt{3}}{4}$	↘	$-\frac{3\sqrt{3}}{4}$	↗	-1

Therefore, **the relative maximum value is $\frac{3\sqrt{3}}{4}$, at $x=\frac{\pi}{6}$** and

the relative minimum value is $-\frac{3\sqrt{3}}{4}$, at $x=\frac{5}{6}\pi$.

The graph is as shown on the right.

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Increasing/Decreasing Functions and Relative Extreme Values

Name ______

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

For each given function, find the relative extreme values, and then draw its graph.

Ex. $y=x\sqrt{1-x^2}$

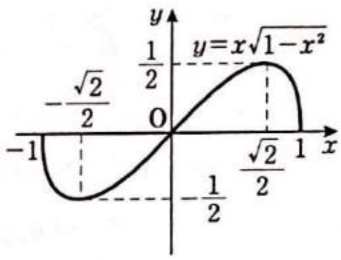
[Sol] Since $1-x^2 \geqslant 0$, the domain is $-1 \leqslant x \leqslant 1$. ⟵ Checking the domain (the expression in the square root $\geqslant 0$)

$$y'=1\cdot\sqrt{1-x^2}+x\cdot\frac{-x}{\sqrt{1-x^2}}=-\frac{2x^2-1}{\sqrt{1-x^2}}$$

When $y'=0$ in $-1<x<1$, $x=\pm\frac{\sqrt{2}}{2}$ ⟵ $2x^2-1=0$

Creating the variation table,

x	-1	$\cdots$	$-\frac{\sqrt{2}}{2}$	$\cdots$	$\frac{\sqrt{2}}{2}$	$\cdots$	1
y'	/	$-$	0	$+$	0	$-$	/
y	0	↘	$-\frac{1}{2}$	↗	$\frac{1}{2}$	↘	0



Therefore, the relative minimum value is $-\frac{1}{2}$, at $x=-\frac{\sqrt{2}}{2}$ and

the relative maximum value is $\frac{1}{2}$, at $x=\frac{\sqrt{2}}{2}$.

The graph is as shown on the right.

(1) $y=x\sqrt{4-x^2}$

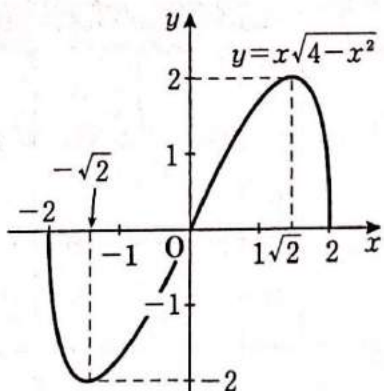
[Sol] Since $4-x^2 \geqslant 0$, the domain is $-2 \leqslant x \leqslant 2$.

$$y'=1\cdot\sqrt{4-x^2}+x\cdot\frac{-x}{\sqrt{4-x^2}}=-\frac{2(x^2-2)}{\sqrt{4-x^2}}$$

When $y'=0$ in $-2<x<2$, $x=\pm\sqrt{2}$

Creating the variation table,

x	-2	$\cdots$	$-\sqrt{2}$	$\cdots$	$\sqrt{2}$	$\cdots$	2
y'	/	$-$	0	$+$	0	$-$	/
y	0	↘	-2	↗	2	↘	0



Therefore, **the relative minimum value is -2, at $x=-\sqrt{2}$** and

the relative maximum value is 2, at $x=\sqrt{2}$.

The graph is as shown on the right.

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O14b

(2) $y=x-2\sqrt{x}$

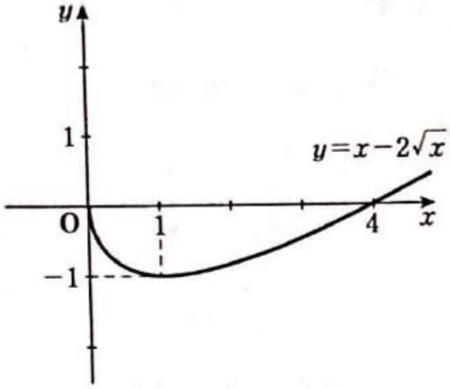
[Sol] The domain is $x \geqslant 0$.

$$y'=1-\frac{1}{\sqrt{x}}=\frac{\sqrt{x}-1}{\sqrt{x}}$$

When $y'=0$ in $x>0$, $x=1$

Creating the variation table,

x	0	$\cdots$	1	$\cdots$
y'		$-$	0	$+$
y	0	$\searrow$	-1	$\nearrow$



Since there are no points where the value of y' switches from $+$ to $-$, there are no relative maximum values.

Therefore, the relative minimum value is $\boxed{-1}$, at $x=\boxed{1}$ and

there are no relative maximum values.

The graph is as shown on the right.

(3) $y=(1-x)\sqrt{x}$

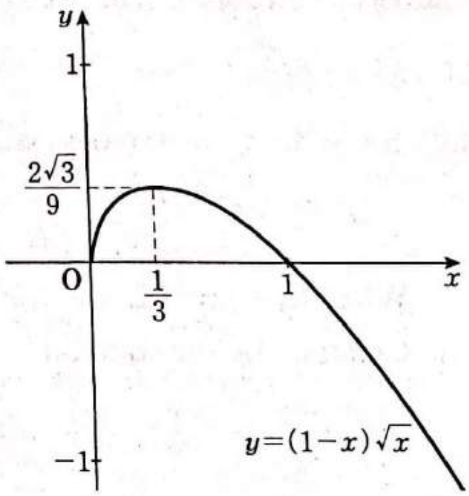
[Sol] The domain is $x \geqslant 0$.

$$y'=-\sqrt{x}+(1-x)\cdot\frac{1}{2\sqrt{x}}=-\frac{3x-1}{2\sqrt{x}}$$

When $y'=0$ in $x>0$, $x=\frac{1}{3}$

Creating the variation table,

x	0	$\cdots$	$\frac{1}{3}$	$\cdots$
y'		$+$	0	$-$
y	0	$\nearrow$	$\frac{2\sqrt{3}}{9}$	$\searrow$



Therefore, **the relative maximum value is $\frac{2\sqrt{3}}{9}$, at $x=\frac{1}{3}$** and

there are no relative minimum values.

The graph is as shown on the right.

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Increasing/Decreasing Functions and Relative Extreme Values

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Name

Date / /

Time : to :

For each given function, find the relative extreme values and the asymptote(s), and then draw its graph.

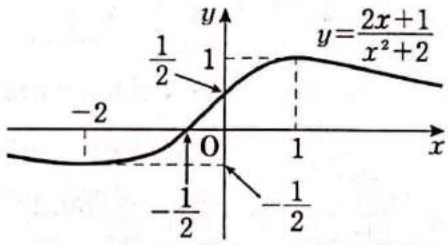
Ex. $y=\dfrac{2x+1}{x^2+2}$

[Sol] $y'=\dfrac{2(x^2+2)-(2x+1)\cdot 2x}{(x^2+2)^2}=-\dfrac{2(x+2)(x-1)}{(x^2+2)^2}$

When $y'=0$, $x=-2,\ 1$

Creating the variation table,

x	$\cdots$	-2	$\cdots$	1	$\cdots$
y'	$-$	0	$+$	0	$-$
y	$\searrow$	$-\dfrac{1}{2}$	$\nearrow$	1	$\searrow$



Therefore, the relative minimum value is $\boxed{-\dfrac{1}{2}}$, at $x=\boxed{-2}$ and

the relative maximum value is $\boxed{1}$, at $x=\boxed{1}$.

Also, since $\lim\limits_{x\to\infty} y=\boxed{0}$ and $\lim\limits_{x\to-\infty} y=\boxed{0}$, ← Since $y=\dfrac{\dfrac{2}{x}+\dfrac{1}{x^2}}{1+\dfrac{2}{x^2}}$

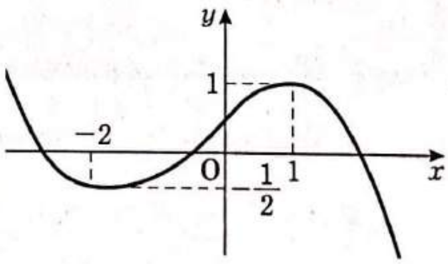
the asymptote is $y=\boxed{0}$.

From the above, the graph is as shown on the right.

Answers: $-\dfrac{1}{2}$, -2, 1, 1, 0, 0, 0

Generally, a line that a graph approaches indefinitely is called an ***asymptote*** of the graph. ← K122

In the case of **Ex.**, if the graph is drawn by referring to the variation table without determining $\lim\limits_{x\to\infty} y$ or $\lim\limits_{x\to-\infty} y$, the graph may have been drawn as shown on the right.



Therefore, when the domain is not a closed interval (for example, all real numbers), it is necessary to determine $\lim\limits_{x\to\infty} y$ or $\lim\limits_{x\to-\infty} y$ as the end values.

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O15b

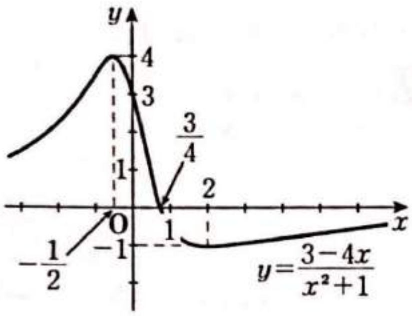
(1) $y=\dfrac{3-4x}{x^2+1}$

[Sol] $y'=\dfrac{-4(x^2+1)-(3-4x)\cdot 2x}{(x^2+1)^2}=\dfrac{2(2x+1)(x-2)}{(x^2+1)^2}$

When $y'=0$, $x=-\dfrac{1}{2}$, 2

Creating the variation table,

x	$\cdots$	$-\dfrac{1}{2}$	$\cdots$	2	$\cdots$
y'	$+$	0	$-$	0	$+$
y	↗	4	↘	-1	↗



Therefore, **the relative maximum value is 4, at $x=-\dfrac{1}{2}$** and

the relative minimum value is -1, at $x=2$.

Also, since $\lim_{x\to\infty} y=0$ and $\lim_{x\to-\infty} y=0$,

the asymptote is $\boldsymbol{y=0}$.

From the above, the graph is as shown on the right.

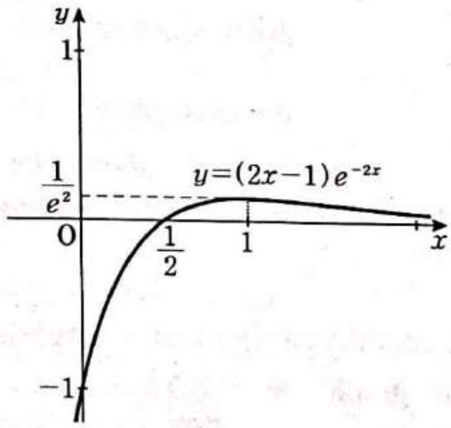
(2) $y=(2x-1)e^{-2x}$ $\left(\lim_{x\to\infty}(2x-1)e^{-2x}=0\right)$

[Sol] $y'=2e^{-2x}+(2x-1)e^{-2x}\cdot(-2)=-4(x-1)e^{-2x}$

When $y'=0$, $x=1$

Creating the variation table,

x	$\cdots$	1	$\cdots$
y'	$+$	0	$-$
y	↗	$\dfrac{1}{e^2}$	↘



Therefore,

the relative maximum value is $\dfrac{1}{e^2}$, at $x=1$ and

there are no relative minimum values.

Also, since $\lim_{x\to\infty} y=0$, ⟵ Since $\lim_{x\to-\infty} y=-\infty$, it does not have to be written.

the asymptote is $\boldsymbol{y=0}$.

From the above, the graph is as shown on the right.

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Increasing/Decreasing Functions and Relative Extreme Values

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
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For each given function, find the relative extreme values and the asymptote(s), and then draw its graph.

Ex. $y=\dfrac{x^2-x+2}{x-2}$

[Sol] The domain is $x\neq 2$. ⟵ Checking the domain (denominator $\neq 0$)

$y=x+1+\dfrac{4}{x-2}$ ⟵

$$\begin{array}{r} x+1 \\ x-2\,\overline{)\,x^2-x+2} \\ \underline{x^2-2x} \\ x+2 \\ \underline{x-2} \\ 4 \end{array}$$

Rearranging can provide a hint to find the asymptotes.

$y'=1-\dfrac{4}{(x-2)^2}=\dfrac{x(x-4)}{(x-2)^2}$

When $y'=0$, $x=0,\ 4$

Creating the variation table,

x	…	0	…	2	…	4	…
y'	+	0	−	/	−	0	+
y	↗	−1	↘	/	↘	7	↗

Therefore, the relative maximum value is $\boxed{-1}$, at $x=\boxed{0}$ and

the relative minimum value is $\boxed{7}$, at $x=\boxed{4}$.

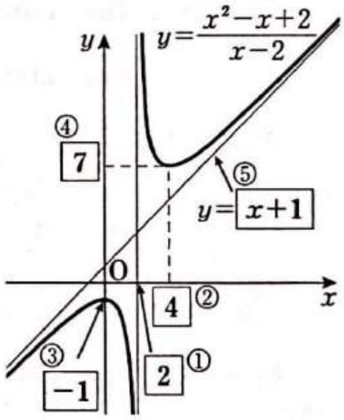
Also, $\lim\limits_{x\to 2^+} y=\boxed{\infty}$, $\lim\limits_{x\to 2^-} y=\boxed{-\infty}$,

$\lim\limits_{x\to\infty}[y-(x+1)]=\boxed{0}$ and

$\lim\limits_{x\to -\infty}[y-(x+1)]=\boxed{0}$. ⟵ Since $y-(x+1)=\dfrac{4}{x-2}$

Thus, the asymptotes are $x=\boxed{2}$ and $y=\boxed{x+1}$.

From the above, the graph is as shown on the right.



Answers:

x	…	0	…	2	…	4	…
y'	+	0	−	/	−	0	+
y	↗	−1	↘	/	↘	7	↗

, −1, 0, 7, 4, ∞, −∞, 0, 0, 2, $x+1$,

(on the graph) ① 2, ② 4, ③ −1, ④ 7, ⑤ $x+1$

The expression can also be rearranged as $\dfrac{x^2-x+2}{x-2}=\dfrac{(x+1)(x-2)+4}{x-2}=x+1+\dfrac{4}{x-2}$.

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016b

(1) $y=\dfrac{x^2}{x+1}$

[Sol] The domain is $x \neq -1$.

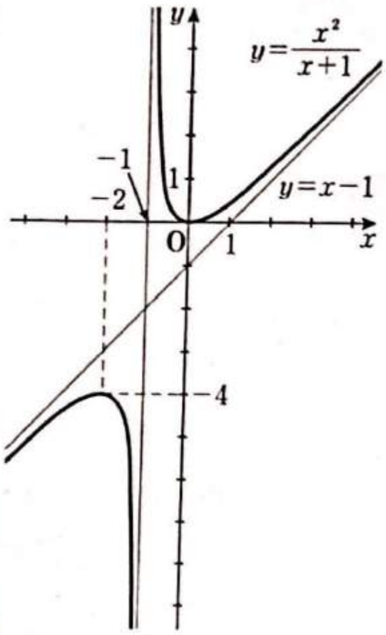
$$y=x-1+\frac{1}{x+1}$$

$$y'=1-\frac{1}{(x+1)^2}=\frac{x(x+2)}{(x+1)^2}$$

When $y'=0$, $x=-2,\ 0$

Creating the variation table,

x	$\cdots$	-2	$\cdots$	-1	$\cdots$	0	$\cdots$
y'	$+$	0	$-$	/	$-$	0	$+$
y	$\nearrow$	-4	$\searrow$	/	$\searrow$	0	$\nearrow$



Therefore, **the relative maximum value is -4, at $x=-2$** and

the relative minimum value is 0, at $x=0$.

Also, $\lim\limits_{x\to-1^+} y=\infty$, $\lim\limits_{x\to-1^-} y=-\infty$,

$$\lim_{x\to\infty}[y-(x-1)]=0 \text{ and } \lim_{x\to-\infty}[y-(x-1)]=0.$$

Thus, the asymptotes are **$x=-1$** and **$y=x-1$.**

From the above, the graph is as shown on the right.

Note Summary

There are two types of asymptotes as follows:

① If $\lim\limits_{x\to\infty}[f(x)-(ax+b)]=0$ or $\lim\limits_{x\to-\infty}[f(x)-(ax+b)]=0$,
line $y=ax+b$ is an asymptote of the curve $y=f(x)$.

② If at least one of $\lim\limits_{x\to a^+} f(x)$ or $\lim\limits_{x\to a^-} f(x)$ is ∞ or $-\infty$,
line $x=a$ is an asymptote of the curve $y=f(x)$.

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Increasing/Decreasing Functions and Relative Extreme Values

Name

Date / /

Time : to :

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For each given function, find the relative extreme values, and then draw its graph.

Ex. $y=|x-3|\sqrt{x}$

[Sol] The domain is $x \geqslant 0$. ← Checking the domain (the expression in the square root $\geqslant 0$)

(i) When $0 \leqslant x < 3$, $y=-(x-3)\sqrt{x}$; therefore,

in $0<x<3$,

$$y'=-\left[1\cdot\sqrt{x}+(x-3)\cdot\frac{1}{2\sqrt{x}}\right]=-\frac{3(\boxed{x-1})}{2\sqrt{x}}$$

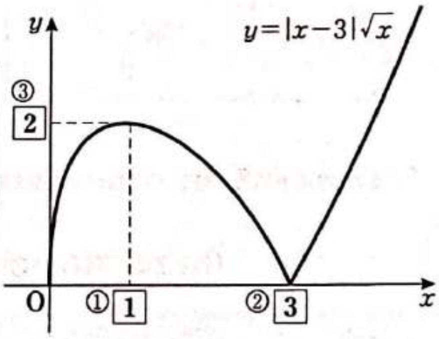
(ii) When $x \geqslant 3$, $y=(x-3)\sqrt{x}$; therefore,

in $x>3$, $y'=\dfrac{3(\boxed{x-1})}{2\sqrt{x}}$

When $y'=0$ in $x>0$, $x=\boxed{1}$

Creating the variation table,

x	0	…	1	…	3	…
y'	/	+	0	−	/	+
y	0	↗	2	↘	0	↗



Therefore, the relative maximum value is $\boxed{2}$, at $x=\boxed{1}$ and

the relative minimum value is $\boxed{0}$, at $x=\boxed{3}$. ←

The graph is as shown on the right. Not differentiable, but having the relative extreme value

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Answers: $x-1$, $x-1$, 1,

x	0	…	1	…	3	…
y'	/	+	0	−	/	+
y	0	↗	2	↘	0	↗

, 2, 1, 0, 3,

(on the graph) ① 1, ② 3, ③ 2

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017b

(1) $y=|x|\sqrt{x+1}$

[Sol] Since $x+1 \geqslant 0$, the domain is $x \geqslant -1$.

(i) When $-1 \leqslant x < 0$, $y=-x\sqrt{x+1}$; therefore,

in $-1 < x < 0$,

$$y'=-\left(1\cdot\sqrt{x+1}+x\cdot\frac{1}{2\sqrt{x+1}}\right)=-\frac{3x+2}{2\sqrt{x+1}}$$

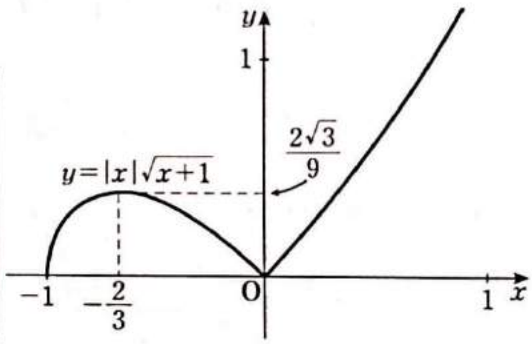
(ii) When $x \geqslant 0$, $y=x\sqrt{x+1}$; therefore,

in $x>0$, $y'=\dfrac{3x+2}{2\sqrt{x+1}}$

When $y'=0$ in $x>-1$, $x=-\dfrac{2}{3}$

Creating the variation table,

x	-1	$\cdots$	$-\frac{2}{3}$	$\cdots$	0	$\cdots$
y'	/	$+$	0	$-$	/	$+$
y	0	$\nearrow$	$\frac{2\sqrt{3}}{9}$	$\searrow$	0	$\nearrow$



Therefore, **the relative maximum value is $\frac{2\sqrt{3}}{9}$, at $x=-\frac{2}{3}$** and

the relative minimum value is 0, at $x=0$.

The graph is as shown on the right.

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Increasing/Decreasing Functions and Relative Extreme Values

100%	~90%	~80%	~70%	69%~
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Name

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Ex.

Find the value of constant a for which function $f(x)=\dfrac{x^2-5x+a}{x-1}$ has a relative extreme value at $x=2$. Then, find the relative extreme values.

[Sol] $f'(x)=\dfrac{(2x-5)(x-1)-(x^2-5x+a)\cdot 1}{(x-1)^2}=\dfrac{x^2-2x+5-a}{(x-1)^2}$

Since $f'(2)=5-a=0$, $a=5$

In order for $f(x)$ to have a relative extreme value at $x=2$, $f'(2)=0$ should be true.

Then, $f(x)=\dfrac{x^2-5x+5}{x-1}$, $f'(x)=\dfrac{x(\boxed{x-2})}{(x-1)^2}$

When $f'(x)=0$, $x=0$, $\boxed{2}$

x	…	0	…	1	…	2	…
$f'(x)$	+	0	−	/	−	0	+
$f(x)$	↗	−5	↘	/	↘	−1	↗

From the variation table, $f(x)$ has a relative extreme value at $x=2$.

$\therefore\ a=5$

Note

When $a=5$, it is necessary to confirm that $f(x)$ has a relative extreme value at $x=2$.

The relative maximum value is $\boxed{-5}$, at $x=\boxed{0}$ and

the relative minimum value is $\boxed{-1}$, at $x=\boxed{2}$.

Answers: $x-2$, 2,

x	…	0	…	1	…	2	…
$f'(x)$	+	0	−	/	−	0	+
$f(x)$	↗	−5	↘	/	↘	−1	↗

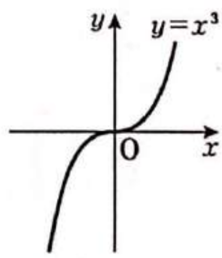
, −5, 0, −1, 2

Note

If $f(x)=x^3$, then $f'(x)=3x^2$.

From the variation table on the right, there are no relative extreme values.

x	…	0	…
$f'(x)$	+	0	+
$f(x)$	↗	0	↗



When a function $f(x)$ is differentiable at $x=a$, and if $f(x)$ has a relative extreme value at $x=a$, $f'(a)=0$ is always satisfied.

However, not all values of a that satisfy $f'(a)=0$ guarantee that $f(x)$ has a relative extreme value at $x=a$. Therefore, it is necessary to confirm that $f(x)$ has a relative extreme value at $x=a$ that satisfies $f'(a)=0$.

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018b

1. Find the values of constants a and b for which function $f(x)=(x^2+ax+b)e^{-x}$ has relative extreme values at $x=-2$ and $x=-1$. Then, find the relative extreme values.

[Sol] $f'(x)=(2x+a)e^{-x}-(x^2+ax+b)e^{-x}$

$=-[x^2+(a-2)x-a+b]e^{-x}$

Since $f'(-2)=(3a-b-8)e^2=0$, $3a-b=8$ …①

Since $f'(-1)=(2a-b-3)e=0$, $2a-b=3$ …②

From ① and ②, $a=5,\ b=7$

Then, $f(x)=(x^2+5x+7)e^{-x}$

$f'(x)=-(x^2+3x+2)e^{-x}$

$=-(x+2)(x+1)e^{-x}$

When $f'(x)=0$, $x=-2,\ -1$

Since $e^{-x}\neq0$

x	$\cdots$	-2	$\cdots$	-1	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	e^2	$\nearrow$	$3e$	$\searrow$

From the variation table, $f(x)$ has relative extreme values at $x=-2$ and $x=-1$.

When $a=5,\ b=7$, it is necessary to confirm that $f(x)$ has relative extreme values at $x=-2$ and $x=-1$.

$\therefore$ $a=5,\ b=7$

The relative minimum value is e^2, at $x=-2$ and

the relative maximum value is $3e$, at $x=-1$.

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Increasing/Decreasing Functions and Relative Extreme Values

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Name

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Time : to :

1. Given that function $f(x)=\dfrac{e^{kx}}{x^2+1}$ $(k>0)$ has relative extreme values, find the range of values of constant k.
(Consider the condition of $f'(x)$ using discriminant D.)

[Sol] $f'(x)=\dfrac{ke^{kx}(x^2+1)-e^{kx}\cdot 2x}{(x^2+1)^2}=\dfrac{(kx^2-2x+k)e^{kx}}{(x^2+1)^2}$

When $f(x)$ has relative extreme values, the sign of $f'(x)$ changes at x where $f'(x)=0$.

Since $x^2+1>0$ and $e^{kx}>0$,

$kx^2-2x+k=0$ should have two different real solutions. ← ※

Therefore, let D be the discriminant of $kx^2-2x+k=0$.

$\dfrac{D}{4}=1-k\cdot k$ ← Discriminant (J122)

$=1-k^2>0$

$\therefore\ k^2<1$

Since $k>0$, $0<k<1$ ← In this case, since the sign of $f'(x)$ changes, it is not necessary to confirm that $f(x)$ has relative extreme values.

※ If there is only one real solution (let that solution be α), the variation table is as shown below. Therefore, since the sign of $f'(x)$ does not change as x increases through α, $f(x)$ has no relative extreme values.

x	$\cdots$	α	$\cdots$
$f'(x)$	$+$	0	$+$
$f(x)$	$\nearrow$	$f(\alpha)$	$\nearrow$

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O19b

2. Let a be a real number. Find the range of values of constant a for which function $f(x)=ax+\cos x+\frac{1}{2}\sin 2x$ does not have any relative extreme values.

[Sol] $f'(x)=a-\sin x+\cos 2x$

$\quad =-2\sin^2 x-\sin x+a+1$ $\quad \boxed{\cos 2\alpha=1-2\sin^2\alpha}$

In order for $f(x)$ not to have any relative extreme values,

the sign of $f'(x)$ should not be changed for all x.

So, $f'(x)\leqslant 0,\ f'(x)\geqslant 0$ have to be true.

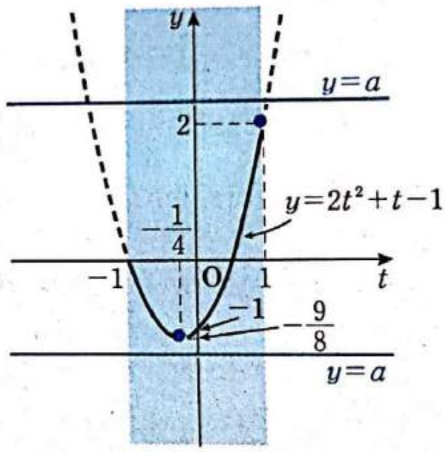
Therefore, $2\sin^2 x+\sin x-1\geqslant a,\ a\geqslant 2\sin^2 x+\sin x-1$

Let $y=2\sin^2 x+\sin x-1$.

Let $\sin x=t$. Then, $-1\leqslant t\leqslant 1$.

$y=2t^2+t-1$

$=2\left(t+\frac{1}{4}\right)^2-\frac{9}{8}$



Thus,

the maximum value is 2, at $t=1$ and

the minimum value is $-\frac{9}{8}$, at $t=-\frac{1}{4}$.

$\therefore\ -\frac{9}{8}\leqslant y\leqslant 2$

$\therefore\ \boldsymbol{a\leqslant -\frac{9}{8},\ 2\leqslant a}$

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Increasing/Decreasing Functions and Relative Extreme Values

Name

Date / /

Time : to :

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1. For the following function, find the relative extreme values, and then draw its graph. ➡ O13

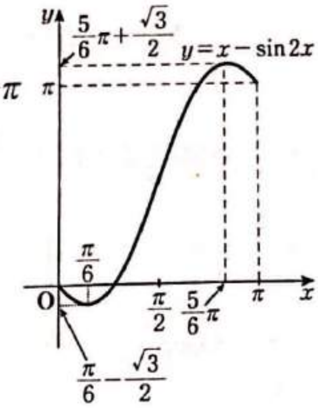
$y=x-\sin 2x \quad (0 \leqq x \leqq \pi)$

[Sol] $y'=1-2\cos 2x$

When $y'=0$ in $0<x<\pi$, $\cos 2x=\frac{1}{2}$, i.e. $x=\frac{\pi}{6}$, $\frac{5}{6}\pi$

Creating the variation table,

x	0	$\cdots$	$\frac{\pi}{6}$	$\cdots$	$\frac{5}{6}\pi$	$\cdots$	π
y'		$-$	0	$+$	0	$-$	
y	0	↘	$\frac{\pi}{6}-\frac{\sqrt{3}}{2}$	↗	$\frac{5}{6}\pi+\frac{\sqrt{3}}{2}$	↘	π



Therefore, **the relative minimum value is $\frac{\pi}{6}-\frac{\sqrt{3}}{2}$, at $x=\frac{\pi}{6}$** and

the relative maximum value is $\frac{5}{6}\pi+\frac{\sqrt{3}}{2}$, at $x=\frac{5}{6}\pi$.

The graph is as shown on the right.

2. For the following function, find the relative extreme values and the asymptote(s), and then draw its graph. ➡ O15

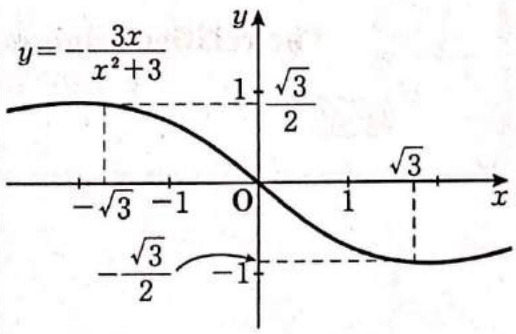
$y=-\frac{3x}{x^2+3}$

[Sol] $y'=-\frac{3(x^2+3)-3x\cdot 2x}{(x^2+3)^2}=\frac{3(x^2-3)}{(x^2+3)^2}$

When $y'=0$, $x=\pm\sqrt{3}$

Creating the variation table,

x	$\cdots$	$-\sqrt{3}$	$\cdots$	$\sqrt{3}$	$\cdots$
y'	$+$	0	$-$	0	$+$
y	↗	$\frac{\sqrt{3}}{2}$	↘	$-\frac{\sqrt{3}}{2}$	↗



Therefore, **the relative maximum value is $\frac{\sqrt{3}}{2}$, at $x=-\sqrt{3}$** and

the relative minimum value is $-\frac{\sqrt{3}}{2}$, at $x=\sqrt{3}$.

Also, since $\lim_{x\to\infty} y=0$ and $\lim_{x\to-\infty} y=0$, the asymptote is $\boldsymbol{y=0}$.

From the above, the graph is as shown on the right.

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O20b

3. Find the values of constants a and b for which function $f(x)=\frac{ax^2+2x+b}{x^2+1}$ has a relative maximum value of 5 at $x=1$. Then, find the relative minimum value.

➡ O18

[Sol] Since $f(1)=\frac{a+2+b}{2}=5$, $a+b=8$ ⋯①

$$f'(x)=\frac{(2ax+2)(x^2+1)-(ax^2+2x+b)\cdot 2x}{(x^2+1)^2}=-\frac{2[x^2-(a-b)x-1]}{(x^2+1)^2}$$

Since $f'(1)=\frac{a-b}{2}=0$, $a-b=0$ ⋯②

From ① and ②, $a=4,\ b=4$

Then, $f(x)=\frac{4x^2+2x+4}{x^2+1}$, $f'(x)=-\frac{2(x^2-1)}{(x^2+1)^2}=-\frac{2(x+1)(x-1)}{(x^2+1)^2}$

When $f'(x)=0$, $x=\pm 1$

x	⋯	-1	⋯	1	⋯
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	↘	3	↗	5	↘

From the variation table,
$f(x)$ has a relative maximum value of 5, at $x=1$.

$\therefore\ a=4,\ b=4$

The relative minimum value is 3, at $x=-1$.

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Concavity of Curves

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Since $f''(x)$ is a derivative of $f'(x)$, the sign of the value of $f''(x)$ can determine whether the value of $f'(x)$ is increasing or decreasing.

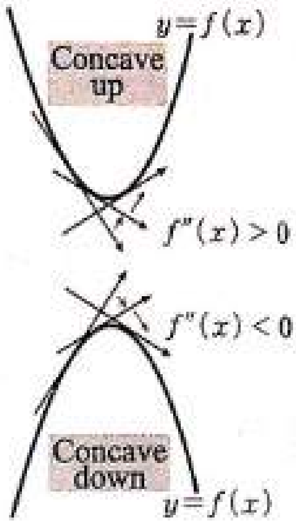
In the interval where $f''(x)>0$, the value of $f'(x)$ is increasing, and the slope of the tangent to the curve $y=f(x)$ is also increasing as shown on the right.

In this case, the curve is said to be ***concave up***.

In the interval where $f''(x)<0$, the value of $f'(x)$ is decreasing and the slope of the tangent to the curve $y=f(x)$ is also decreasing as shown on the right.

In this case, the curve is said to be ***concave down***.

Then, the following statement is true.



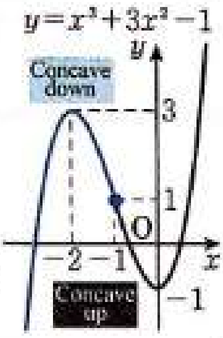
Concavity of Curves

When a function $f(x)$ has the second order derivative $f''(x)$,
in the interval where $f''(x)>0$, the curve $y=f(x)$ is concave up; and
in the interval where $f''(x)<0$, the curve $y=f(x)$ is concave down.

1. Determine the concavity of each given curve.

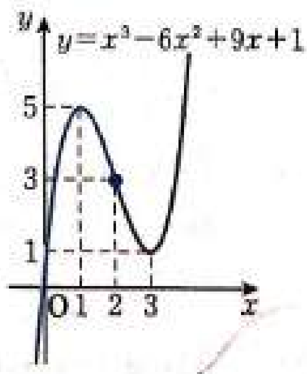
Ex. $y=x^3+3x^2-1$

[Sol] $y'=3x^2+6x,\ y''=6x+6=6(x+1)$
When $y''=0$, $x=-1$
When $x<-1$, $y''<0$, and when $x>-1$, $y''>0$
Therefore, the curve is concave down when $x<-1$, and concave up when $x>-1$.



(1) $y=x^3-6x^2+9x+1$

[Sol] $y'=3x^2-12x+9,\ y''=6x-12=6(x-2)$
When $y''=0$, $x=2$
When $x<2$, $y''<0$, and when $x>2$, $y''>0$
Therefore, the curve is **concave down when $x<2$**, and **concave up when $x>2$.**

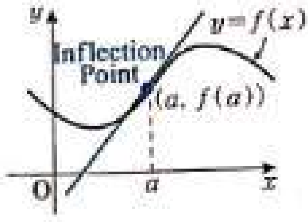


021b

The curve in **Ex.** on side a changes its concavity at point $(-1, 1)$.
The point where the curve $y=f(x)$ changes its concavity is called an ***inflection point*** or a ***point of inflection***.

Inflection Point

Given $f''(a)=0$, if the sign of $f''(x)$ changes as x increases through a, point $(a, f(a))$ is an inflection point of the curve $y=f(x)$. Also, if point $(a, f(a))$ is an inflection point of the curve $y=f(x)$, then $f''(a)=0$.



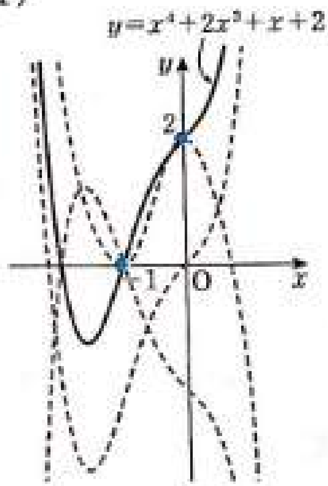
2. For each given curve, determine the concavity and find the inflection point(s). Then, trace the corresponding graph.

Ex. $y=x^4+2x^3+x+2$

[Sol] $y'=4x^3+6x^2+1$, $y''=12x^2+12x=12x(x+1)$
When $y''=0$, $x=-1, 0$

x	$\cdots$	-1	$\cdots$	0	$\cdots$
y''	$+$	0	$-$	0	$+$
y	Concave up	0	Concave down	2	Concave up

From the variation table, the curve is concave up when $x<-1$, $0<x$, and concave down when $-1<x<0$.
The inflection points are $(-1, 0)$, $(0, 2)$.
The graph is as shown on the right.

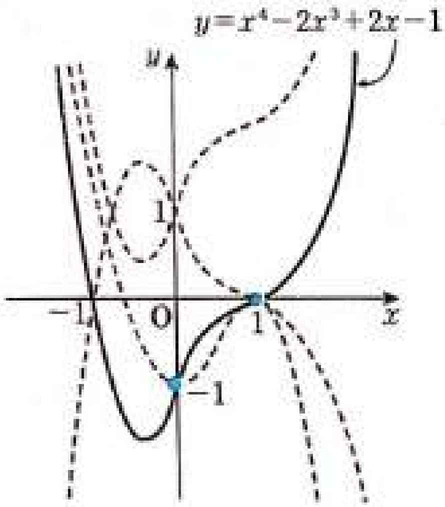


(1) $y=x^4-2x^3+2x-1$

[Sol] $y'=4x^3-6x^2+2$, $y''=12x^2-12x=12x(x-1)$
When $y''=0$, $x=0, 1$

x	$\cdots$	0	$\cdots$	1	$\cdots$
y''	$+$	0	$-$	0	$+$
y	Concave up	-1	Concave down	0	Concave up

From the variation table, the curve is
concave up when $x<0$, $1<x$, and
concave down when $0<x<1$.
The inflection points are **$(0, -1)$, $(1, 0)$.**
The graph is as shown on the right.



Concavity of Curves

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For each given curve, determine the concavity and find the inflection point(s). Then, trace the corresponding graph.

(1) $y=x+\cos 2x \quad (0<x<\pi)$

[Sol] $y'=1-2\sin 2x,\ y''=-4\cos 2x$

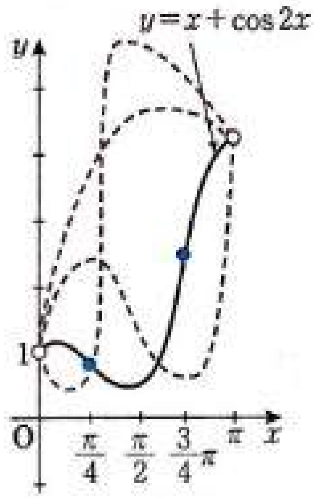
When $y''=0$ in $0<x<\pi$, $\cos 2x=0$, i.e. $x=\frac{\pi}{4},\ \frac{3}{4}\pi$

x	0	$\cdots$	$\frac{\pi}{4}$	$\cdots$	$\frac{3}{4}\pi$	$\cdots$	π
y''	/	$-$	0	$+$	0	$-$	/
y	/	Concave down	$\frac{\pi}{4}$	Concave up	$\frac{3}{4}\pi$	Concave down	/

From the variation table, the curve is **concave down when $0<x<\frac{\pi}{4},\ \frac{3}{4}\pi<x<\pi$**, and **concave up when $\frac{\pi}{4}<x<\frac{3}{4}\pi$.**

The inflection points are $\left(\frac{\pi}{4}, \frac{\pi}{4}\right), \left(\frac{3}{4}\pi, \frac{3}{4}\pi\right)$.

The graph is as shown on the right.



(2) $y=\frac{4}{x^2+3}$

[Sol] $y'=-\frac{4\cdot 2x}{(x^2+3)^2}=-\frac{8x}{(x^2+3)^2}$

$$y''=-\frac{8(x^2+3)^2-8x\cdot 2(x^2+3)\cdot 2x}{(x^2+3)^4}=\frac{24(x+1)(x-1)}{(x^2+3)^3}$$

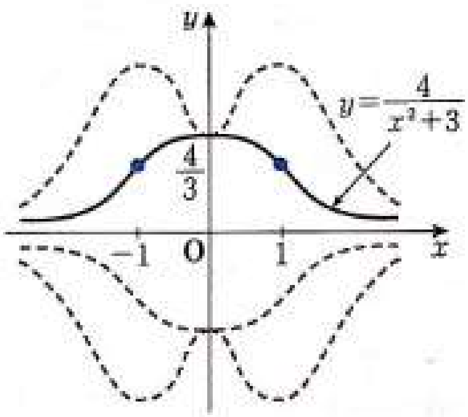
When $y''=0$, $x=\pm 1$

x	$\cdots$	-1	$\cdots$	1	$\cdots$
y''	$+$	0	$-$	0	$+$
y	Concave up	1	Concave down	1	Concave up

From the variation table, the curve is **concave up when $x<-1,\ 1<x$**, and **concave down when $-1<x<1$.**

The inflection points are $(-1, 1), (1, 1)$.

The graph is as shown on the right.



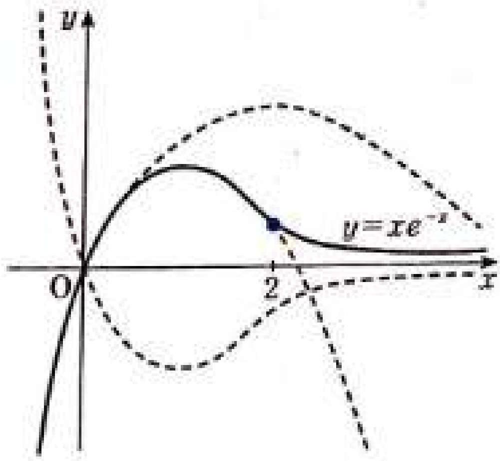
O22b

(3) $y=xe^{-x}$

[Sol] $y'=1\cdot e^{-x}-xe^{-x}=(1-x)e^{-x}$, $y''=-e^{-x}-(1-x)e^{-x}=-(2-x)e^{-x}$

When $y''=0$, $x=2$

x	$\cdots$	2	$\cdots$
y''	$-$	0	$+$
y	Concave down	$\frac{2}{e^2}$	Concave up



From the variation table, the curve is **concave down when $x < 2$**, and **concave up when $x > 2$.**

The inflection point is $\left(2, \frac{2}{e^2}\right)$.

The graph is as shown on the right.

(4) $y=x^2\ln x$

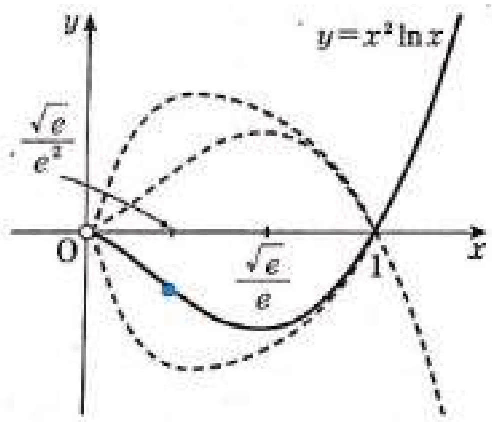
[Sol] The domain is $x > 0$. ⟵ Antilogarithm > 0

$y'=2x\ln x+x^2\cdot\frac{1}{x}=x(2\ln x+1)$, $y''=1\cdot(2\ln x+1)+x\cdot\frac{2}{x}=2\ln x+3$

When $y''=0$ in $x > 0$,

$\ln x=-\frac{3}{2}$, i.e. $x=\frac{\sqrt{e}}{e^2}$ ⟵ $x=e^{-\frac{3}{2}}=\frac{1}{e\sqrt{e}}=\frac{\sqrt{e}}{e^2}$

x	0	$\cdots$	$\frac{\sqrt{e}}{e^2}$	$\cdots$
y''	/	$-$	0	$+$
y	/	Concave down	$-\frac{3}{2e^3}$	Concave up



From the variation table, the curve is

concave down when $0 < x < \frac{\sqrt{e}}{e^2}$, and

concave up when $x > \frac{\sqrt{e}}{e^2}$.

The inflection point is $\left(\frac{\sqrt{e}}{e^2}, -\frac{3}{2e^3}\right)$. $\left[\frac{\sqrt{e}}{e^2}=\frac{1}{e\sqrt{e}}\right]$

The graph is as shown on the right.

Concavity of Curves

Name

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For each given function, determine where the function increases/decreases, its relative extreme values, concavity and inflection point(s). Then, draw its graph.

Ex. $y=x^4-4x^3$

[Sol] $y'=4x^3-12x^2=4x^2(x-3)$

$y''=12x^2-24x=12x(x-2)$

When $y'=0$, $x=0,\ 3$. When $y''=0$, $x=0,\ 2$.

Creating the variation table,

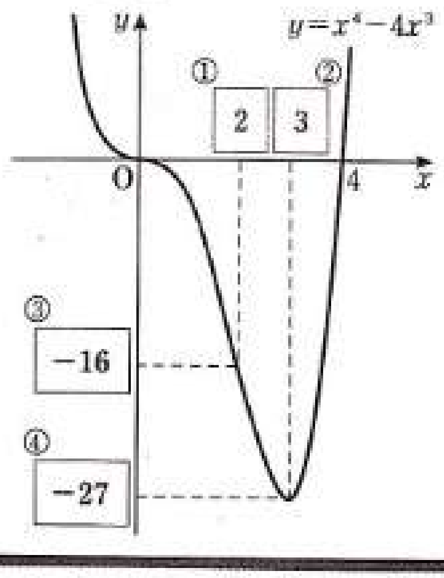
x	$\cdots$	0	$\cdots$	2	$\cdots$	3	$\cdots$
y'	$-$	0	$-$	$-$	$-$	0	$+$
y''	$+$	0	$-$	0	$+$	$+$	$+$
y	↘	0	↘	-16	↘	-27	↗

↘ means concave up and decreasing (y' is $-$ and y'' is $+$).
↗ means concave up and increasing (y' is $+$ and y'' is $+$).
↗ means concave down and increasing (y' is $+$ and y'' is $-$).
↘ means concave down and decreasing (y' is $-$ and y'' is $-$).

Therefore, the relative minimum value is $\boxed{-27}$, at $x=\boxed{3}$ and there are no relative maximum values.

The inflection points are $(0, \boxed{0})$, $(\boxed{2}, \boxed{-16})$.

From the above, the graph is as shown below.



When $x=0$, $y'=0$. However, since the sign of y' does not change as x increases through 0, this function does not have the relative extreme values at $x=0$.

Answers: -27, 3, 0, 2, -16, (on the graph) ① 2, ② 3, ③ -16, ④ -27

By finding the second order derivative as shown in **Ex.**, it is possible to sketch the graph more in detail.

O23b

(1) $y=2x^4-4x^3+3$

[Sol] $y'=8x^3-12x^2=4x^2(2x-3)$

$y''=24x^2-24x=24x(x-1)$

When $y'=0$, $x=0,\ \frac{3}{2}$. When $y''=0$, $x=0,\ 1$.

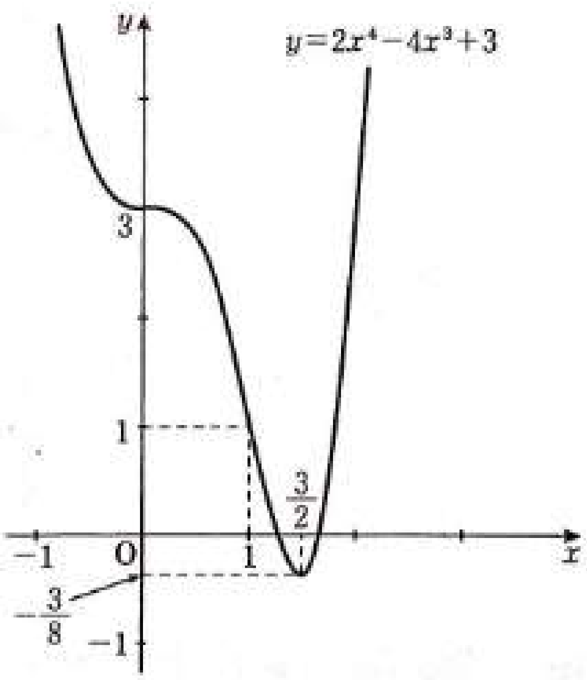
Creating the variation table,

x	$\cdots$	0	$\cdots$	1	$\cdots$	$\frac{3}{2}$	$\cdots$
y'	$-$	0	$-$	$-$	$-$	0	$+$
y''	$+$	0	$-$	0	$+$	$+$	$+$
y	↘	3	↘	1	↘	$-\frac{3}{8}$	↗

Therefore, **the relative minimum value is $-\frac{3}{8}$, at $x=\frac{3}{2}$** and **there are no relative maximum values.**

The inflection points are **$(0, 3)$, $(1, 1)$.**

From the above, the graph is as shown below.



Concavity of Curves

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For each given function, determine where the function increases/decreases, its relative extreme values, concavity and inflection point(s). Then, draw its graph.

(1) $y=x-2\sin x \quad (0 \leqq x \leqq 2\pi)$

[Sol] $y'=1-2\cos x$

$y''=2\sin x$

In $0 < x < 2\pi$, ⟵ Considering $y'=0$ and $y''=0$ on the open interval

when $y'=0$, $\cos x=\dfrac{1}{2}$, i.e. $x=\dfrac{\pi}{3}$, $\dfrac{5}{3}\pi$; and

when $y''=0$, $\sin x=0$, i.e. $x=\pi$

Creating the variation table,

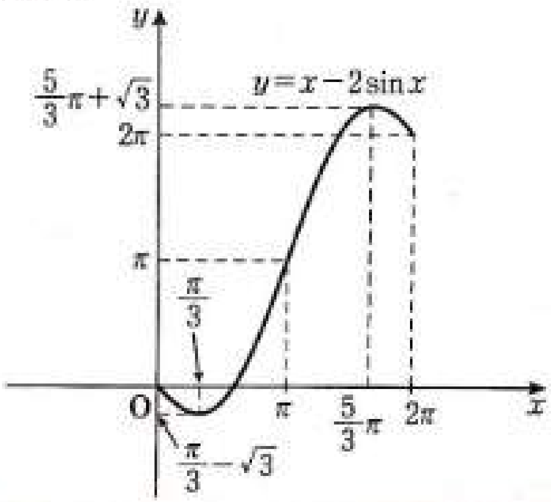
x	0	$\cdots$	$\frac{\pi}{3}$	$\cdots$	π	$\cdots$	$\frac{5}{3}\pi$	$\cdots$	2π
y'		$-$	0	$+$	$+$	$+$	0	$-$	
y''		$+$	$+$	$+$	0	$-$	$-$	$-$	
y	0	↘	$\frac{\pi}{3}-\sqrt{3}$	↗	π	↗	$\frac{5}{3}\pi+\sqrt{3}$	↘	2π

Therefore, **the relative minimum value is $\dfrac{\pi}{3}-\sqrt{3}$, at $x=\dfrac{\pi}{3}$** and

the relative maximum value is $\dfrac{5}{3}\pi+\sqrt{3}$, at $x=\dfrac{5}{3}\pi$.

The inflection point is (π, π).

From the above, the graph is as shown below.



O24b

(2) $y=4\cos x+\cos 2x \quad (0 \leqq x \leqq 2\pi)$

[Sol] $y'=-4\sin x-2\sin 2x$

$=-4\sin x-4\sin x\cos x$

$=-4\sin x(1+\cos x)$

$\sin 2\alpha=2\sin\alpha\cos\alpha$

$y''=-4\cos x-4\cos 2x$

$=-4(2\cos^2 x+\cos x-1)$

$=-4(\cos x+1)(2\cos x-1)$

In this step, $y'=-4\sin x-2\sin 2x$ is differentiated, but it is also possible to differentiate $y'=-4\sin x(1+\cos x)$.

In $0<x<2\pi$,

when $y'=0$, $\sin x=0$, $\cos x=-1$, i.e. $x=\pi$; and

when $y''=0$, $\cos x=-1,\ \frac{1}{2}$, i.e. $x=\frac{\pi}{3},\ \pi,\ \frac{5}{3}\pi$

Creating the variation table,

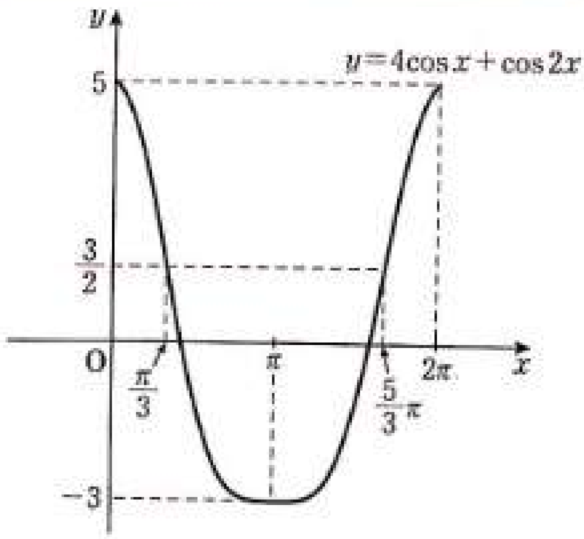
x	0	$\cdots$	$\frac{\pi}{3}$	$\cdots$	π	$\cdots$	$\frac{5}{3}\pi$	$\cdots$	2π
y'		$-$	$-$	$-$	0	$+$	$+$	$+$	
y''		$-$	0	$+$	0	$+$	0	$-$	
y	5	↘	$\frac{3}{2}$	↘	-3	↗	$\frac{3}{2}$	↗	5

Therefore, **the relative minimum value is -3, at $x=\pi$** and **there are no relative maximum values.**

The inflection points are $\left(\frac{\pi}{3},\frac{3}{2}\right)$, $\left(\frac{5}{3}\pi,\frac{3}{2}\right)$.

From the above, the graph is as shown below.

When $x=\pi$, $y''=0$. However, since the sign of y'' does not change as x increases through π, $(\pi, -3)$ is not the inflection point.



Concavity of Curves

Name

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Time : to :

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(mistakes) 0	—	—	—	1~

For each given function, determine where the function increases/decreases, its relative extreme values, concavity, inflection point(s) and asymptote(s). Then, draw its graph.

Ex.

$y=\dfrac{1}{x^2+1}$

[Sol] $y'=-\dfrac{2x}{(x^2+1)^2}$

$$y''=-\frac{2(x^2+1)^2-2x\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4}=\frac{2(3x^2-1)}{(x^2+1)^3}$$

When $y'=0$, $x=0$. When $y''=0$, $x=\pm\dfrac{\sqrt{3}}{3}$.

Creating the variation table,

x	$\cdots$	$-\frac{\sqrt{3}}{3}$	$\cdots$	0	$\cdots$	$\frac{\sqrt{3}}{3}$	$\cdots$
y'	$+$	$+$	$+$	0	$-$	$-$	$-$
y''	$+$	0	$-$	$-$	$-$	0	$+$
y	↗	$\frac{3}{4}$	↗	1	↘	$\frac{3}{4}$	↘

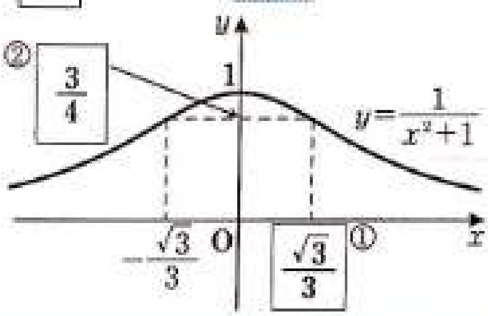
Therefore, the relative maximum value is $\boxed{1}$, at $x=\boxed{0}$ and there are no relative minimum values.

The inflection points are $\left(-\dfrac{\sqrt{3}}{3}, \boxed{\dfrac{3}{4}}\right)$, $\left(\boxed{\dfrac{\sqrt{3}}{3}}, \boxed{\dfrac{3}{4}}\right)$.

Also, since $\lim\limits_{x\to\infty} y=\boxed{0}$ and $\lim\limits_{x\to-\infty} y=\boxed{0}$, ← O15

the asymptote is $y=\boxed{0}$.

From the above, the graph is as shown on the right.



Answers: 1, 0, $\frac{3}{4}$, $\frac{\sqrt{3}}{3}$, $\frac{3}{4}$, 0, 0, 0, (on the graph) ① $\frac{\sqrt{3}}{3}$, ② $\frac{3}{4}$

O25b

(1) $y=\dfrac{2x}{x^2+1}$

[Sol] $y'=\dfrac{2(x^2+1)-2x\cdot 2x}{(x^2+1)^2}=-\dfrac{2(x+1)(x-1)}{(x^2+1)^2}$

$$y''=-\frac{4x(x^2+1)^2-2(x^2-1)\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4}=\frac{4x(x^2-3)}{(x^2+1)^3}$$

When $y'=0$, $x=\pm 1$. When $y''=0$, $x=0,\ \pm\sqrt{3}$.

Creating the variation table,

x	$\cdots$	$-\sqrt{3}$	$\cdots$	-1	$\cdots$	0	$\cdots$	1	$\cdots$	$\sqrt{3}$	$\cdots$
y'	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$
y''	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
y	↘	$-\frac{\sqrt{3}}{2}$	↘	-1	↗	0	↗	1	↘	$\frac{\sqrt{3}}{2}$	↘

Therefore, **the relative minimum value is -1, at $x=-1$** and

the relative maximum value is 1, at $x=1$.

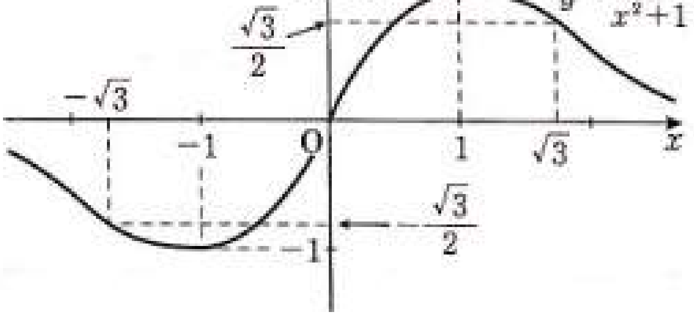
The inflection points are $\left(-\sqrt{3},\ -\dfrac{\sqrt{3}}{2}\right)$, $(0,\ 0)$, $\left(\sqrt{3},\ \dfrac{\sqrt{3}}{2}\right)$.

Also, since $\lim\limits_{x\to\infty} y=0$ and $\lim\limits_{x\to-\infty} y=0$,

the asymptote is **$y=0$.**

Since $y=\dfrac{\frac{2}{x}}{1+\frac{1}{x^2}}$

From the above, the graph is as shown below.



Concavity of Curves

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For each given function, determine where the function increases/decreases, its relative extreme values, concavity, inflection point(s) and asymptote(s). Then, draw its graph.

(1) $y=e^{-2x^2}$

[Sol] $y'=e^{-2x^2}\cdot(-4x)=-4xe^{-2x^2}$

$y''=-4[1\cdot e^{-2x^2}+xe^{-2x^2}\cdot(-4x)]=4(2x+1)(2x-1)e^{-2x^2}$

When $y'=0$, $x=0$. When $y''=0$, $x=\pm\frac{1}{2}$.

Creating the variation table,

x	$\cdots$	$-\frac{1}{2}$	$\cdots$	0	$\cdots$	$\frac{1}{2}$	$\cdots$
y'	$+$	$+$	$+$	0	$-$	$-$	$-$
y''	$+$	0	$-$	$-$	$-$	0	$+$
y	⤴	$\frac{\sqrt{e}}{e}$	↗	1	↘	$\frac{\sqrt{e}}{e}$	⤵

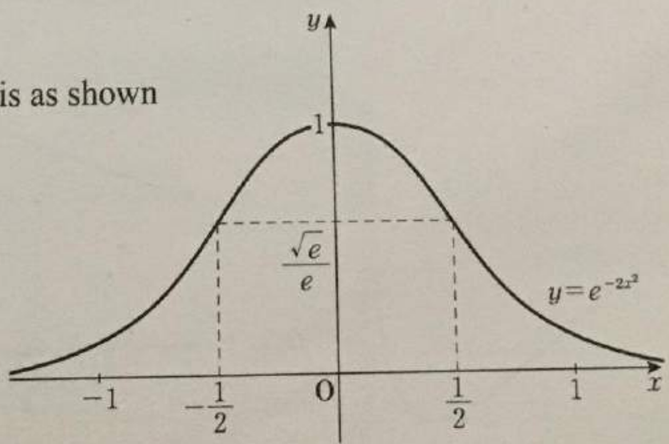
Therefore, **the relative maximum value is 1, at $x=0$** and **there are no relative minimum values.**

The inflection points are $\left(-\frac{1}{2}, \frac{\sqrt{e}}{e}\right)$, $\left(\frac{1}{2}, \frac{\sqrt{e}}{e}\right)$. $\left[\frac{\sqrt{e}}{e}=\frac{1}{\sqrt{e}}\right]$

Also, since $\lim_{x\to\infty} y=0$ and $\lim_{x\to-\infty} y=0$,

the asymptote is $\boldsymbol{y=0}$.

From the above, the graph is as shown on the right.



O26b

(2) $y=\dfrac{1}{\sqrt{x^2+1}}$

[Sol] $y'=-\dfrac{\dfrac{x}{\sqrt{x^2+1}}}{x^2+1}=-\dfrac{x}{(x^2+1)\sqrt{x^2+1}}$

Since $y=(x^2+1)^{-\frac{1}{2}}$, it is also possible to calculate as $y'=-\frac{1}{2}(x^2+1)^{-\frac{3}{2}}\cdot 2x$.

$$y''=-\frac{1\cdot(x^2+1)\sqrt{x^2+1}-x\cdot\frac{3}{2}\sqrt{x^2+1}\cdot 2x}{(x^2+1)^3}=\frac{2x^2-1}{(x^2+1)^2\sqrt{x^2+1}}$$

When $y'=0$, $x=0$. When $y''=0$, $x=\pm\dfrac{\sqrt{2}}{2}$.

Creating the variation table,

x	$\cdots$	$-\frac{\sqrt{2}}{2}$	$\cdots$	0	$\cdots$	$\frac{\sqrt{2}}{2}$	$\cdots$
y'	$+$	$+$	$+$	0	$-$	$-$	$-$
y''	$+$	0	$-$	$-$	$-$	0	$+$
y	↗	$\frac{\sqrt{6}}{3}$	↗	1	↘	$\frac{\sqrt{6}}{3}$	↘

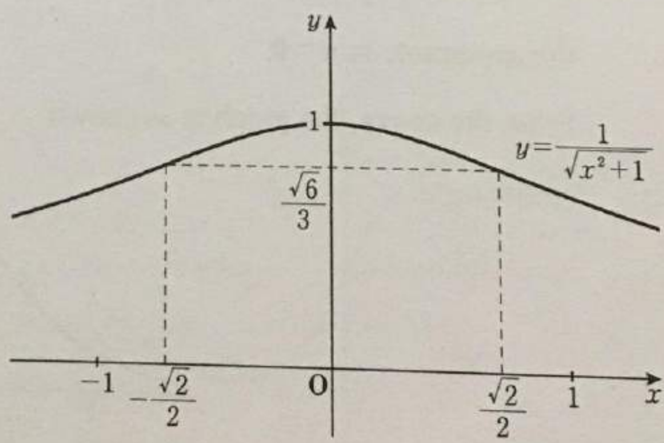
Therefore, **the relative maximum value is 1, at $x=0$** and **there are no relative minimum values.**

The inflection points are $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{6}}{3}\right)$, $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{6}}{3}\right)$.

Also, since $\lim\limits_{x\to\infty} y=0$ and $\lim\limits_{x\to-\infty} y=0$,

the asymptote is $\boldsymbol{y=0}$.

From the above, the graph is as shown below.



Concavity of Curves

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For each given function, determine where the function increases/decreases, its relative extreme values, concavity, inflection point(s) and asymptote(s). Then, draw its graph.

(1) $y=(\ln x)^2$

[Sol] The domain is $x>0$. ⟵ Antilogarithm >0

$$y'=2\ln x\cdot\frac{1}{x}=\frac{2\ln x}{x}$$

$$y''=\frac{\frac{2}{x}\cdot x-2\ln x\cdot 1}{x^2}=\frac{2(1-\ln x)}{x^2}$$

In $x>0$,

when $y'=0$, $\ln x=0$, i.e. $x=1$; and

when $y''=0$, $\ln x=1$, i.e. $x=e$

Creating the variation table,

x	0	$\cdots$	1	$\cdots$	e	$\cdots$
y'		$-$	0	$+$	$+$	$+$
y''		$+$	$+$	$+$	0	$-$
y		↘	0	↗	1	↗

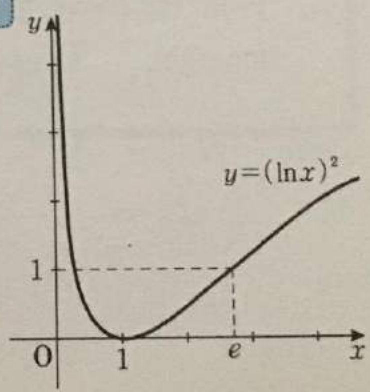
Therefore, **the relative minimum value is 0, at $x=1$** and **there are no relative maximum values.**

The inflection point is **$(e, 1)$**.

Also, since $\lim_{x\to 0^+} y=\infty$, ⟵ Since $\lim_{x\to 0^+}\ln x=-\infty$

the asymptote is **$x=0$**.

From the above, the graph is as shown on the right.



O27b

(2) $y=\dfrac{\ln x}{x}$ $\left(\lim\limits_{x\to\infty}\dfrac{\ln x}{x}=0\right)$

[Sol] The domain is $x>0$.

$$y'=\frac{\frac{1}{x}\cdot x-\ln x\cdot 1}{x^2}=\frac{1-\ln x}{x^2}$$

$$y''=\frac{-\frac{1}{x}\cdot x^2-(1-\ln x)\cdot 2x}{x^4}=\frac{2\ln x-3}{x^3}$$

In $x>0$,

when $y'=0$, $\ln x=1$, i.e. $x=e$; and

when $y''=0$, $\ln x=\dfrac{3}{2}$, i.e. $x=e\sqrt{e}$

Creating the variation table,

x	0	$\cdots$	e	$\cdots$	$e\sqrt{e}$	$\cdots$
y'	╱	$+$	0	$-$	$-$	$-$
y''	╱	$-$	$-$	$-$	0	$+$
y	╱	↗	$\dfrac{1}{e}$	↘	$\dfrac{3\sqrt{e}}{2e^2}$	↘

Therefore, **the relative maximum value is $\dfrac{1}{e}$, at $x=e$** and

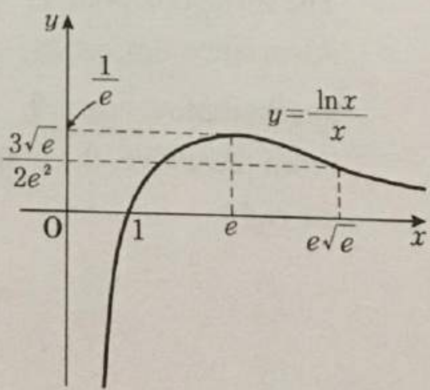
there are no relative minimum values.

The inflection point is $\left(e\sqrt{e},\ \dfrac{3\sqrt{e}}{2e^2}\right)$. $\left[\dfrac{3\sqrt{e}}{2e^2}=\dfrac{3}{2e\sqrt{e}}\right]$

Also, since $\lim\limits_{x\to 0^+}y=-\infty$ and $\lim\limits_{x\to\infty}y=0$,

the asymptotes are $x=0,\ y=0$.

From the above, the graph is as shown on the right.



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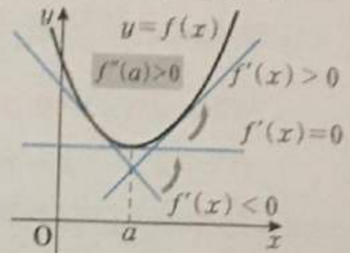
The relative extreme values can also be determined by the sign of the second order derivative, not only by the variation table.



When a function $f(x)$ satisfies $f'(a)=0$ and $f''(a)>0$, $f'(x)$ is increasing at the point where $x=a$ or close to $x=a$, and $f'(x)$ changes from negative to positive as x increases through a.

Therefore, $f(x)$ has the relative minimum value at $x=a$.

This method can also be used for the relative maximum value, giving the statements below.

Sign of $f''(a)$ and Relative Extreme Values

Given that $f''(x)$ is continuous on the interval including $x=a$,

if $f'(a)=0$ and $f''(a)>0$, $f(a)$ represents the relative minimum value;

if $f'(a)=0$ and $f''(a)<0$, $f(a)$ represents the relative maximum value.

For each given function, find the relative extreme values by using the second order derivative.

Ex. $f(x)=2x^3-6x+3$

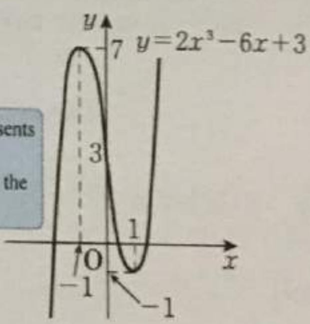
[Sol] $f'(x)=6x^2-6=6(x+1)(x-1)$

$f''(x)=\boxed{12x}$

When $f'(x)=0$, $x=\pm 1$

$f''(-1)=\boxed{-12}<0$, $f''(1)=\boxed{12}>0$

Since $f''(-1)<0$, $f(-1)$ represents the relative maximum value.
Since $f''(1)>0$, $f(1)$ represents the relative minimum value.



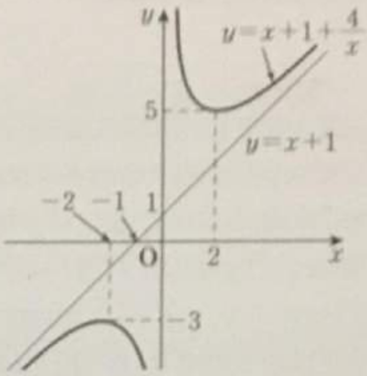
Therefore, the relative maximum value is $f(-1)=\boxed{7}$, and

the relative minimum value is $f(1)=\boxed{-1}$.

Answers: $12x$, -12, 12, 7, -1

O28b

(1) $f(x)=x+1+\frac{4}{x}$



[Sol] $f'(x)=1-\frac{4}{x^2}=\frac{(x+2)(x-2)}{x^2}$

$f''(x)=\frac{4\cdot 2x}{x^4}=\frac{8}{x^3}$

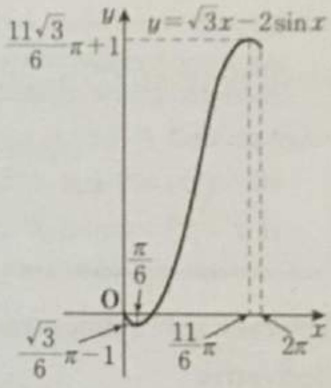
When $f'(x)=0$, $x=\pm 2$

$f''(-2)=-1<0$, $f''(2)=1>0$

Therefore, **the relative maximum value is $f(-2)=-3$,** and

the relative minimum value is $f(2)=5$.

(2) $f(x)=\sqrt{3}x-2\sin x \quad (0 \leqq x \leqq 2\pi)$



[Sol] $f'(x)=\sqrt{3}-2\cos x$

$f''(x)=2\sin x$

When $f'(x)=0$ in $0<x<2\pi$,

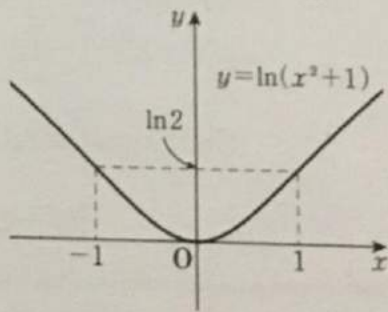
$\cos x=\frac{\sqrt{3}}{2}$, i.e. $x=\frac{\pi}{6}$, $\frac{11}{6}\pi$

$f''\left(\frac{\pi}{6}\right)=1>0$, $f''\left(\frac{11}{6}\pi\right)=-1<0$

Therefore, **the relative minimum value is $f\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{6}\pi-1$,** and

the relative maximum value is $f\left(\frac{11}{6}\pi\right)=\frac{11\sqrt{3}}{6}\pi+1$.

(3) $f(x)=\ln(x^2+1)$



[Sol] $f'(x)=\frac{2x}{x^2+1}$

$f''(x)=\frac{2(x^2+1)-2x\cdot 2x}{(x^2+1)^2}=-\frac{2(x^2-1)}{(x^2+1)^2}$

When $f'(x)=0$, $x=0$

$f''(0)=2>0$

Therefore, **the relative minimum value is $f(0)=0$,** and

there are no relative maximum values.

KUMON

Concavity of Curves

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1. Given that curve $y=x^4+ax^3+3ax^2+1$ has inflection point(s), find the range of values of constant a. (Consider the condition of y'' using discriminant D.)

[Sol] $y'=4x^3+3ax^2+6ax$

$y''=12x^2+6ax+6a=6(2x^2+ax+a)$

When this curve has inflection points, the sign of y'' changes at x where $y''=0$.

Therefore, $2x^2+ax+a=0$ should have two different real solutions.

Thus, let D be the discriminant of $2x^2+ax+a=0$.

$$D=a^2-4\cdot2\cdot a$$

Discriminant (J122)

$$=a^2-8a$$
$$=a(a-8)>0$$
$$\therefore\ a<0,\ 8<a$$

If there is only one real solution (let that solution be α), the variation table is as below. Therefore, since the sign of y'' does not change as x increases through α, there are no inflection points.

x	$\cdots$	α	$\cdots$
y''	$+$	0	$+$
y	Concave up		Concave up

2. Given that a is a constant, find the number of inflection points of curve $y=(x^2+2x+a)e^x$.

[Sol] $y'=(2x+2)e^x+(x^2+2x+a)e^x=(x^2+4x+a+2)e^x$

$y''=(2x+4)e^x+(x^2+4x+a+2)e^x=(x^2+6x+a+6)e^x$

When this curve has inflection points, the sign of y'' changes at x where $y''=0$.

Since $e^x>0$, $x^2+6x+a+6=0$ should have two different real solutions.

Therefore, let D be the discriminant of $x^2+6x+a+6=0$.

$$\frac{D}{4}=3^2-1\cdot(a+6)$$
$$=-a+3$$

Thus, this curve has

2 inflection points when $a<3$, and

no inflection points when $a \geqslant 3$.

Let α and β be the two different real solutions of $x^2+6x+a+6=0$ $(\alpha<\beta)$. The variation table is as below, showing that there are 2 inflection points.

x	$\cdots$	α	$\cdots$	β	$\cdots$
y''	$+$	0	$-$	0	$+$
y	Concave up	Inflection point	Concave down	Inflection point	Concave up

029b

3. Draw the graph of $y^2=x^2(x+1)$.

[Sol] Since $y^2 \geqq 0$, $x^2(x+1) \geqq 0$; therefore, the domain is $x \geqq -1$.

Also, since $y^2=x^2(x+1)$ is true when y is replaced by $-y$,

this graph is symmetric with respect to the x-axis.

Since $y=\pm x\sqrt{x+1}$, consider curve $y=x\sqrt{x+1}$.

$$y'=1\cdot\sqrt{x+1}+x\cdot\frac{1}{2\sqrt{x+1}}=\frac{3x+2}{2\sqrt{x+1}}$$

The graph to be found will be the one which $y=x\sqrt{x+1}$ and $y=-x\sqrt{x+1}$ are combined.

$$y''=\frac{3\cdot 2\sqrt{x+1}-(3x+2)\cdot\frac{1}{\sqrt{x+1}}}{4(x+1)}=\frac{3x+4}{4(x+1)\sqrt{x+1}}$$

$x=-\frac{4}{3}$ is not within the domain.

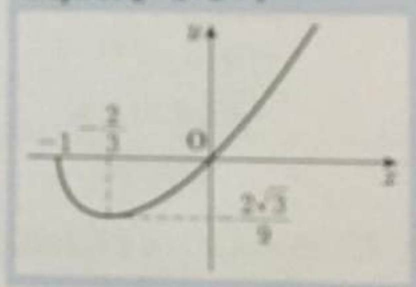
In $x>-1$,

when $y'=0$, $x=-\frac{2}{3}$, and there is no x that satisfies $y''=0$.

Creating the variation table,

x	-1	$\cdots$	$-\frac{2}{3}$	$\cdots$
y'	/	$-$	0	$+$
y''	/	$+$	$+$	$+$
y	0	↘	$-\frac{2\sqrt{3}}{9}$	↗

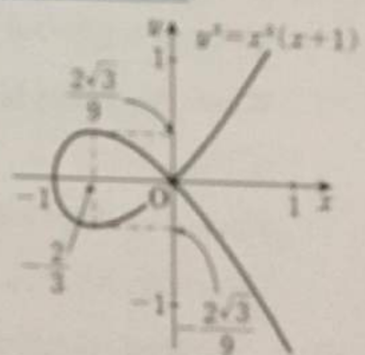
Graph of $y=x\sqrt{x+1}$



Also, $\lim_{x\to -1+} y'=-\infty$

It shows that the curve is tangent to line $x=-1$.

Since the graph of $y=-x\sqrt{x+1}$ is symmetric to $y=x\sqrt{x+1}$ with respect to the x-axis, the graph is as shown on the right.



(Pay attention to the shape of the graph near point $(-1, 0)$.)

If the equation is the same even after y is replaced by $-y$, the graph is symmetric with respect to the x-axis.

Likewise, if the equation is the same even after x is replaced by

Concavity of Curves

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1. For the following function, determine where the function increases/decreases, its relative extreme values, concavity and inflection point(s). Then, draw its graph.

$y=3x^4-4x^3$

➡ O23

[Sol] $y'=12x^3-12x^2=12x^2(x-1)$

$y''=36x^2-24x=12x(3x-2)$

When $y'=0$, $x=0,\ 1$. When $y''=0$, $x=0,\ \frac{2}{3}$.

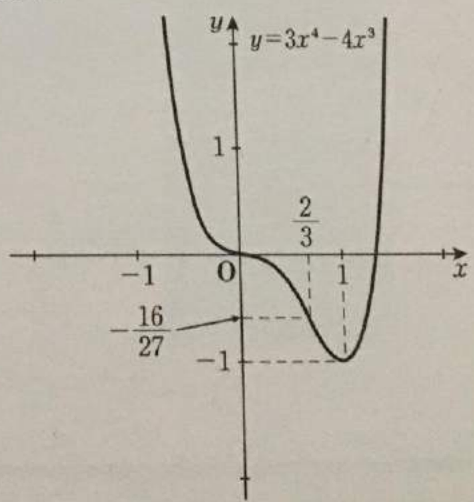
Creating the variation table,

x	$\cdots$	0	$\cdots$	$\frac{2}{3}$	$\cdots$	1	$\cdots$
y'	$-$	0	$-$	$-$	$-$	0	$+$
y''	$+$	0	$-$	0	$+$	$+$	$+$
y	↘	0	↘	$-\frac{16}{27}$	↘	-1	↗

Therefore, **the relative minimum value is -1, at $x=1$** and **there are no relative maximum values.**

The inflection points are $(0,\ 0)$, $\left(\frac{2}{3},\ -\frac{16}{27}\right)$.

From the above, the graph is as shown below.



O30b

2. For the following function, determine where the function increases/decreases, its relative extreme values, concavity, inflection point(s) and asymptote(s). Then, draw its graph.
➡ O25

$y=xe^x \quad \left(\lim_{x\to-\infty} xe^x=0\right)$

[Sol] $y'=1\cdot e^x+xe^x=(x+1)e^x$

$y''=1\cdot e^x+(x+1)e^x=(x+2)e^x$

When $y'=0$, $x=-1$. When $y''=0$, $x=-2$.

Creating the variation table,

x	$\cdots$	-2	$\cdots$	-1	$\cdots$
y'	$-$	$-$	$-$	0	$+$
y''	$-$	0	$+$	$+$	$+$
y	↘	$-\frac{2}{e^2}$	↘	$-\frac{1}{e}$	↗

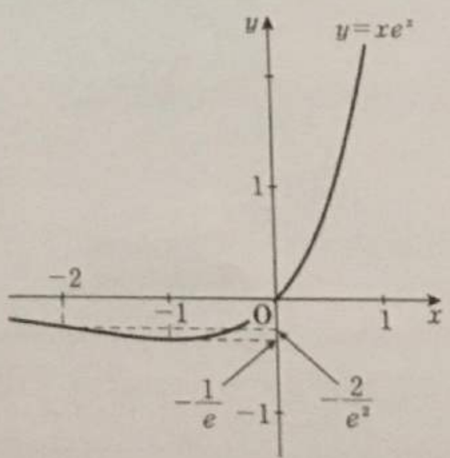
Therefore, **the relative minimum value is $-\frac{1}{e}$, at $x=-1$** and **there are no relative maximum values.**

The inflection point is $\left(-2, -\frac{2}{e^2}\right)$.

Also, since $\lim_{x\to-\infty} y=0$,

the asymptote is $y=0$.

From the above, the graph is as shown below.



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Find the maximum and minimum values of each given function.

Ex. $y=x-2\sin x \quad (0 \leqq x \leqq \pi)$

[Sol] $y'=1-2\cos x$

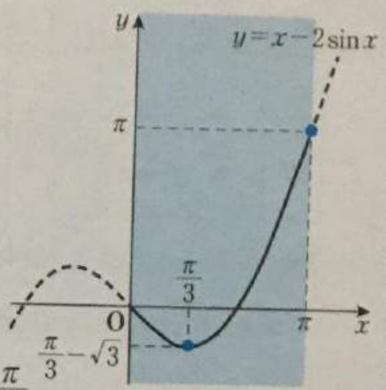
When $y'=0$ in $0<x<\pi$, $\cos x=\frac{1}{2}$, i.e. $x=\frac{\pi}{3}$

x	0	$\cdots$	$\frac{\pi}{3}$	$\cdots$	π
y'		$-$	0	$+$	
y	0	↘	$\frac{\pi}{3}-\sqrt{3}$	↗	π

From the variation table,

the maximum value is π, at $x=\pi$ and

the minimum value is $\frac{\pi}{3}-\sqrt{3}$, at $x=\frac{\pi}{3}$.



(1) $y=x+2\cos x \quad \left(-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}\right)$

[Sol] $y'=1-2\sin x$

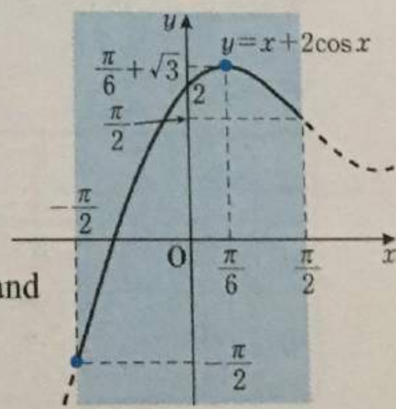
When $y'=0$ in $-\frac{\pi}{2}<x<\frac{\pi}{2}$, $\sin x=\frac{1}{2}$, i.e. $x=\frac{\pi}{6}$

x	$-\frac{\pi}{2}$	$\cdots$	$\frac{\pi}{6}$	$\cdots$	$\frac{\pi}{2}$
y'		$+$	0	$-$	
y	$-\frac{\pi}{2}$	↗	$\frac{\pi}{6}+\sqrt{3}$	↘	$\frac{\pi}{2}$

From the variation table,

the maximum value is $\frac{\pi}{6}+\sqrt{3}$, at $x=\frac{\pi}{6}$ and

the minimum value is $-\frac{\pi}{2}$, at $x=-\frac{\pi}{2}$.



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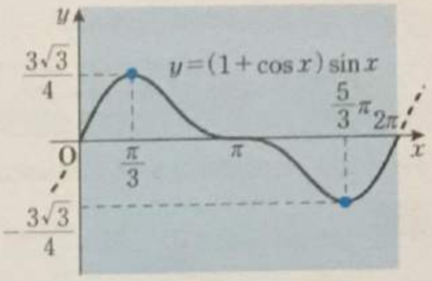
031b

(2) $y=(1+\cos x)\sin x \quad (0 \leqq x \leqq 2\pi)$



[Sol] $y'=-\sin x\cdot\sin x+(1+\cos x)\cos x$

$=-\sin^2 x+\cos x+\cos^2 x$

$=2\cos^2 x+\cos x-1$

$=(\cos x+1)(2\cos x-1)$

When $y'=0$ in $0<x<2\pi$,

$\cos x=-1,\ \frac{1}{2}$, i.e. $x=\frac{\pi}{3},\ \pi,\ \frac{5}{3}\pi$

x	0	$\cdots$	$\frac{\pi}{3}$	$\cdots$	π	$\cdots$	$\frac{5}{3}\pi$	$\cdots$	2π
y'		$+$	0	$-$	0	$-$	0	$+$	
y	0	↗	$\frac{3\sqrt{3}}{4}$	↘	0	↘	$-\frac{3\sqrt{3}}{4}$	↗	0

From the variation table,

the maximum value is $\frac{3\sqrt{3}}{4}$, at $x=\frac{\pi}{3}$ and

the minimum value is $-\frac{3\sqrt{3}}{4}$, at $x=\frac{5}{3}\pi$.

(3) $y=\sin^3 x+\cos^3 x \quad (0 \leqq x \leqq \pi)$

[Sol] $y'=3\sin^2 x\cos x+3\cos^2 x\cdot(-\sin x)$

$=3\sin x\cos x(\sin x-\cos x)$

$=3\sqrt{2}\sin x\cos x\sin\left(x-\frac{\pi}{4}\right)$

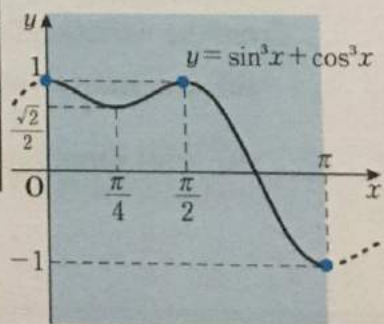
$a\sin\theta+b\cos\theta=\sqrt{a^2+b^2}\sin(\theta+\alpha)$

When $y'=0$ in $0<x<\pi$, $\cos x=0$, $\sin\left(x-\frac{\pi}{4}\right)=0$

When $0<x<\pi$, $\sin x \neq 0$

i.e. $x=\frac{\pi}{4},\ \frac{\pi}{2}$

x	0	$\cdots$	$\frac{\pi}{4}$	$\cdots$	$\frac{\pi}{2}$	$\cdots$	π
y'		$-$	0	$+$	0	$-$	
y	1	↘	$\frac{\sqrt{2}}{2}$	↗	1	↘	-1



From the variation table,

the maximum value is 1, at $x=0$, $\frac{\pi}{2}$ and

the minimum value is -1, at $x=\pi$.

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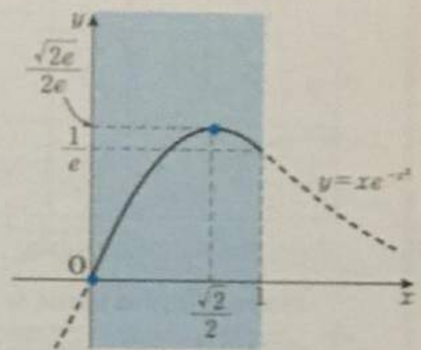
Find the maximum and minimum values of each given function.

(1) $y=xe^{-x^2}$ $(0 \leqslant x \leqslant 1)$

[Sol] $y'=1\cdot e^{-x^2}+xe^{-x^2}\cdot(-2x)=(1-2x^2)e^{-x^2}$

When $y'=0$ in $0<x<1$, $x=\frac{\sqrt{2}}{2}$

x	0	$\cdots$	$\frac{\sqrt{2}}{2}$	$\cdots$	1
y'		$+$	0	$-$	
y	0	$\nearrow$	$\frac{\sqrt{2e}}{2e}$	$\searrow$	$\frac{1}{e}$



From the variation table,

the maximum value is $\frac{\sqrt{2e}}{2e}$ $\left[\frac{\sqrt{2e}}{2e}=\frac{1}{\sqrt{2e}}\right]$, at $x=\frac{\sqrt{2}}{2}$ and

the minimum value is 0, at $x=0$.

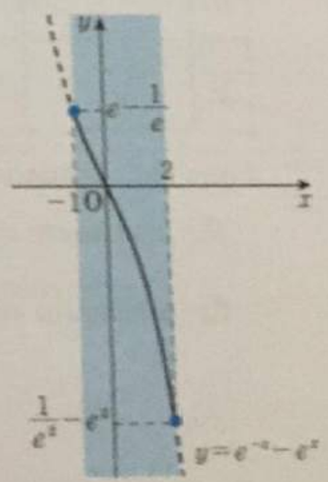
(2) $y=e^{-x}-e^{x}$ $(-1 \leqslant x \leqslant 2)$

Since $e^{-x}>0$ and $e^{x}>0$,
$y'=-(e^{-x}+e^{x})<0$

[Sol] $y'=-e^{-x}-e^{x}$

In $-1<x<2$, there is no x that satisfies $y'=0$.

x	-1	$\cdots$	2
y'		$-$	
y	$e-\frac{1}{e}$	$\searrow$	$\frac{1}{e^2}-e^2$



From the variation table,

the maximum value is $e-\frac{1}{e}$, at $x=-1$ and

the minimum value is $\frac{1}{e^2}-e^2$, at $x=2$.

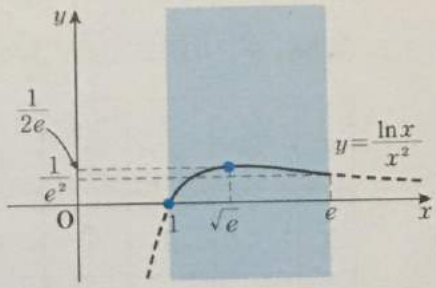
O32b

(3) $y=\dfrac{\ln x}{x^2}\quad(1\leqslant x\leqslant e)$

[Sol] $y'=\dfrac{\dfrac{1}{x}\cdot x^2-\ln x\cdot 2x}{x^4}=\dfrac{1-2\ln x}{x^3}$

When $y'=0$ in $1<x<e$, $\ln x=\dfrac{1}{2}$, i.e. $x=\sqrt{e}$

x	1	$\cdots$	$\sqrt{e}$	$\cdots$	e
y'		$+$	0	$-$	
y	0	$\nearrow$	$\dfrac{1}{2e}$	$\searrow$	$\dfrac{1}{e^2}$



From the variation table,

the maximum value is $\dfrac{1}{2e}$, at $x=\sqrt{e}$ and

the minimum value is 0, at $x=1$.

(4) $y=\dfrac{x}{x^2+4}\quad(-3\leqslant x\leqslant 3)$

[Sol] $y'=\dfrac{1\cdot(x^2+4)-x\cdot 2x}{(x^2+4)^2}=-\dfrac{(x+2)(x-2)}{(x^2+4)^2}$

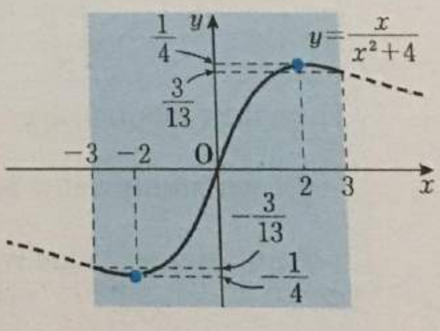
When $y'=0$ in $-3<x<3$, $x=\pm 2$

x	-3	$\cdots$	-2	$\cdots$	2	$\cdots$	3
y'		$-$	0	$+$	0	$-$	
y	$-\dfrac{3}{13}$	$\searrow$	$-\dfrac{1}{4}$	$\nearrow$	$\dfrac{1}{4}$	$\searrow$	$\dfrac{3}{13}$

From the variation table,

the maximum value is $\dfrac{1}{4}$, at $x=2$ and

the minimum value is $-\dfrac{1}{4}$, at $x=-2$.



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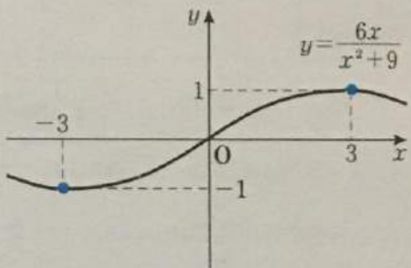
Find the maximum and minimum values of each given function.

Ex. $y=\dfrac{6x}{x^2+9}$

[Sol] $y'=\dfrac{6(x^2+9)-6x\cdot 2x}{(x^2+9)^2}=-\dfrac{6(x+3)(x-3)}{(x^2+9)^2}$

When $y'=0$, $x=\pm 3$

x	$\cdots$	-3	$\cdots$	3	$\cdots$
y'	$-$	0	$+$	0	$-$
y	$\searrow$	-1	$\nearrow$	1	$\searrow$



Also, since $\lim\limits_{x\to\infty} y=0$ and $\lim\limits_{x\to-\infty} y=0$, from the variation table,

the maximum value is 1, at $x=3$ and

the minimum value is -1, at $x=-3$.

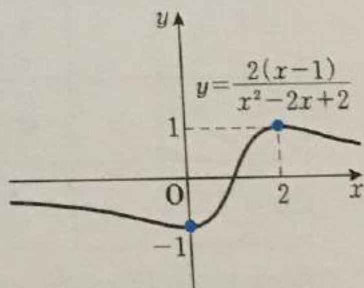
If the domain is all real numbers, determine $\lim\limits_{x\to\infty} y$ and $\lim\limits_{x\to-\infty} y$ as the values on both ends.

(1) $y=\dfrac{2(x-1)}{x^2-2x+2}$

[Sol] $y'=\dfrac{2(x^2-2x+2)-2(x-1)(2x-2)}{(x^2-2x+2)^2}=-\dfrac{2x(x-2)}{(x^2-2x+2)^2}$

When $y'=0$, $x=0,\ 2$

x	$\cdots$	0	$\cdots$	2	$\cdots$
y'	$-$	0	$+$	0	$-$
y	$\searrow$	-1	$\nearrow$	1	$\searrow$



Also, since $\lim\limits_{x\to\infty} y=0$ and $\lim\limits_{x\to-\infty} y=0$, from the variation table,

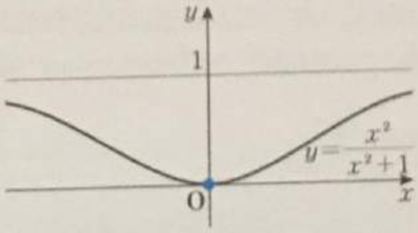
the maximum value is 1, at $x=2$ and

the minimum value is -1, at $x=0$.

O33b

(2) $y=\dfrac{x^2}{x^2+1}$

[Sol] $y'=\dfrac{2x(x^2+1)-x^2\cdot 2x}{(x^2+1)^2}=\dfrac{2x}{(x^2+1)^2}$



When $y'=0$, $x=0$

x	$\cdots$	0	$\cdots$
y'	$-$	0	$+$
y	$\searrow$	0	$\nearrow$

Since $y=\dfrac{1}{1+\dfrac{1}{x^2}}$

Also, since $\lim\limits_{x\to\infty} y=\boxed{1}$ and $\lim\limits_{x\to-\infty} y=\boxed{1}$, from the variation table,

there are no maximum values, and

the minimum value is $\boxed{0}$, at $x=\boxed{0}$.

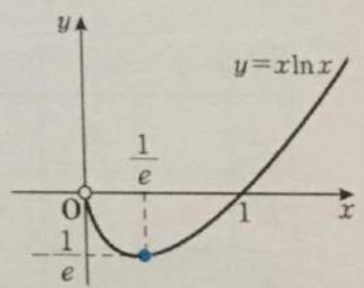
If the domain is not a closed interval, a maximum value or minimum value does not always exist.

(3) $y=x\ln x$ $\left(\lim\limits_{x\to 0^+} x\ln x=0\right)$

[Sol] The domain is $x>0$.

$y'=1\cdot\ln x+x\cdot\dfrac{1}{x}=\ln x+1$

When $y'=0$ in $x>0$, $\ln x=-1$, i.e. $x=\dfrac{1}{e}$



x	0	$\cdots$	$\dfrac{1}{e}$	$\cdots$
y'	/	$-$	0	$+$
y	/	$\searrow$	$-\dfrac{1}{e}$	$\nearrow$

Also, since $\lim\limits_{x\to 0^+} y=0$ and $\lim\limits_{x\to\infty} y=\infty$, from the variation table,

there are no maximum values, and

the minimum value is $-\dfrac{1}{e}$, at $x=\dfrac{1}{e}$.

Since the domain is $x>0$, determine $\lim\limits_{x\to 0^+} y$ and $\lim\limits_{x\to\infty} y$ as the values on both ends.

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Find the maximum and minimum values of each given function.

Ex. $y=x+\sqrt{4-x^2}$

[Sol] Since $4-x^2 \geqslant 0$, the domain is $-2 \leqslant x \leqslant 2$.

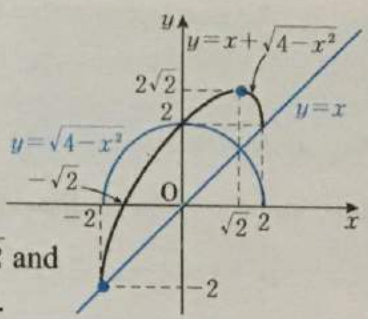
$$y'=1-\frac{x}{\sqrt{4-x^2}}=\frac{\sqrt{4-x^2}-x}{\sqrt{4-x^2}}$$

When $y'=0$ in $-2<x<2$, $\sqrt{4-x^2}=x$,

i.e. $x=\sqrt{2}$

Since $4-x^2=x^2$,
$x^2=2$
Since $\sqrt{4-x^2}=x$,
$x \geqslant 0$
$\therefore\ x=\sqrt{2}$

x	-2	$\cdots$	$\sqrt{2}$	$\cdots$	2
y'		$+$	0	$-$	
y	-2	$\nearrow$	$2\sqrt{2}$	$\searrow$	2



From the variation table,
the maximum value is $2\sqrt{2}$, at $x=\sqrt{2}$ and
the minimum value is -2, at $x=-2$.

(1) $y=x+\sqrt{1-x^2}$

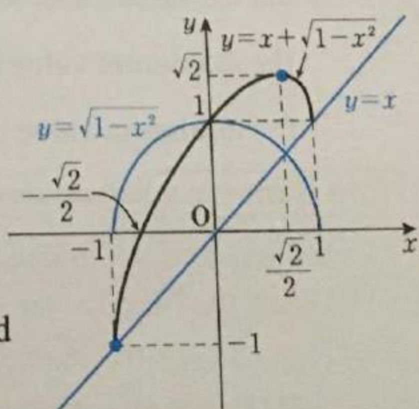
[Sol] Since $1-x^2 \geqslant 0$, the domain is $-1 \leqslant x \leqslant 1$.

$$y'=1-\frac{x}{\sqrt{1-x^2}}=\frac{\sqrt{1-x^2}-x}{\sqrt{1-x^2}}$$

When $y'=0$ in $-1<x<1$, $\sqrt{1-x^2}=x$,

i.e. $x=\frac{\sqrt{2}}{2}$

x	-1	$\cdots$	$\frac{\sqrt{2}}{2}$	$\cdots$	1
y'		$+$	0	$-$	
y	-1	$\nearrow$	$\sqrt{2}$	$\searrow$	1



From the variation table,

the maximum value is $\sqrt{2}$, at $x=\frac{\sqrt{2}}{2}$ and

the minimum value is -1, at $x=-1$.

O34b

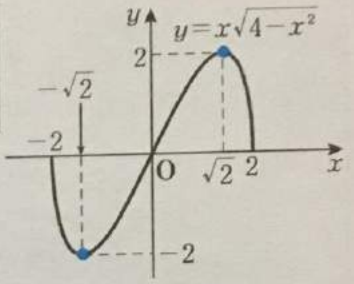
(2) $y=x\sqrt{4-x^2}$

[Sol] Since $4-x^2 \geqslant 0$, the domain is $-2 \leqslant x \leqslant 2$.

$$y'=1\cdot\sqrt{4-x^2}+x\cdot\frac{-x}{\sqrt{4-x^2}}=-\frac{2(x^2-2)}{\sqrt{4-x^2}}$$

When $y'=0$ in $-2<x<2$, $x=\pm\sqrt{2}$

x	-2	$\cdots$	$-\sqrt{2}$	$\cdots$	$\sqrt{2}$	$\cdots$	2
y'	/	$-$	0	$+$	0	$-$	/
y	0	↘	-2	↗	2	↘	0



From the variation table,

the maximum value is 2, at $x=\sqrt{2}$ and
the minimum value is -2, at $x=-\sqrt{2}$.

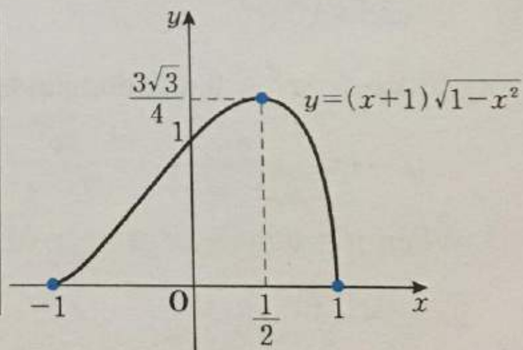
(3) $y=(x+1)\sqrt{1-x^2}$

[Sol] Since $1-x^2 \geqslant 0$, the domain is $-1 \leqslant x \leqslant 1$.

$$y'=1\cdot\sqrt{1-x^2}+(x+1)\cdot\frac{-x}{\sqrt{1-x^2}}=-\frac{(x+1)(2x-1)}{\sqrt{1-x^2}}$$

When $y'=0$ in $-1<x<1$, $x=\frac{1}{2}$

x	-1	$\cdots$	$\frac{1}{2}$	$\cdots$	1
y'	/	$+$	0	$-$	/
y	0	↗	$\frac{3\sqrt{3}}{4}$	↘	0



From the variation table,

the maximum value is $\frac{3\sqrt{3}}{4}$, at $x=\frac{1}{2}$ and
the minimum value is 0, at $x=\pm1$.

Note Summary

To find the maximum and minimum values:

① Create the variation table for the given interval or domain.

② Find the relative extreme values and the values on both ends of the interval, and then compare the values.

Maxima and Minima

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Ex. Find a for which the minimum value of function $y=x+\dfrac{a}{x-1}\ (x>1)$ is 5. $(a>0)$

[Sol] $y'=1-\dfrac{a}{(x-1)^2}=\dfrac{x^2-2x+1-a}{(x-1)^2}$

When $y'=0$ in $x>1$, $x=1+\sqrt{a}$

Solving quadratic equation $x^2-2x+1-a=0$, $x=1\pm\sqrt{a}$. Since $x>1$, $x=1+\sqrt{a}$

x	1	$\cdots$	$1+\sqrt{a}$	$\cdots$
y'		$-$	0	$+$
y		$\searrow$	$1+2\sqrt{a}$	$\nearrow$

Since the minimum value can be defined as $1+2\sqrt{a}$ from the variation table, $\lim\limits_{x\to 0^+} y$ and $\lim\limits_{x\to\infty} y$ do not have to be determined.

From the variation table,
the minimum value is $1+2\sqrt{a}$, at $x=1+\sqrt{a}$.
Since this value is equal to 5, $1+2\sqrt{a}=5$

$\therefore\ a=4$

1. Find a for which the minimum value of function $y=x\ln x+a$ is $3a+2$.

[Sol] The domain is $x>0$.

$y'=1\cdot\ln x+x\cdot\dfrac{1}{x}=\ln x+1$

When $y'=0$ in $x>0$, $\ln x=-1$, i.e. $x=\dfrac{1}{e}$

x	0	$\cdots$	$\dfrac{1}{e}$	$\cdots$
y'		$-$	0	$+$
y		$\searrow$	$-\dfrac{1}{e}+a$	$\nearrow$

From the variation table, the minimum value is $-\dfrac{1}{e}+a$, at $x=\dfrac{1}{e}$.

Since this value is equal to $3a+2$, $-\dfrac{1}{e}+a=3a+2$

$\therefore\ a=-\dfrac{1}{2e}-1$

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O35b

2. Find a for which the maximum value of function $f(x)=\dfrac{a\sin x}{\cos x+2}$ $(0 \leqq x \leqq \pi)$ is $\sqrt{3}$.

[Sol] $f'(x)=\dfrac{a\cos x(\cos x+2)-a\sin x\cdot(-\sin x)}{(\cos x+2)^2}=\dfrac{a(\cos^2 x+2\cos x+\sin^2 x)}{(\cos x+2)^2}=\dfrac{a(2\cos x+1)}{(\cos x+2)^2}$

The sign of $f'(x)$ changes depending on the value of a. Therefore, different conditions should be considered.

(i) When $a=0$, $f(x)=0$ for all values of x.

Therefore, it is not applicable.

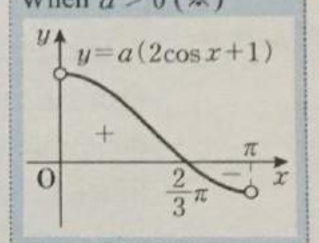
(ii) When $a>0$

When $f'(x)=0$ in $0<x<\pi$, $\cos x=-\dfrac{1}{2}$,

i.e. $x=\dfrac{2}{3}\pi$

x	0	$\cdots$	$\frac{2}{3}\pi$	$\cdots$	π
$f'(x)$		$+$	0	$-$	
$f(x)$	0	$\nearrow$	$\frac{\sqrt{3}}{3}a$	$\searrow$	0

When $a>0$ (※)



From the variation table, the maximum value is $\dfrac{\sqrt{3}}{3}a$, at $x=\dfrac{2}{3}\pi$.

Since this value is equal to $\sqrt{3}$, $\dfrac{\sqrt{3}}{3}a=\sqrt{3}$

$\therefore\ a=3$

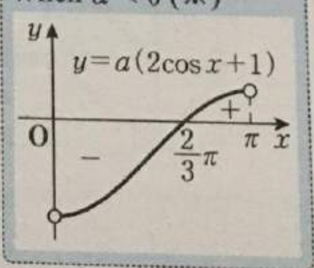
This satisfies $a>0$.

※ Since $(\cos x+2)^2>0$ when $0 \leqq x \leqq \pi$, the sign of $f'(x)$ is the same as the sign of $y=a(2\cos x+1)$.

(iii) When $a<0$

x	0	$\cdots$	$\frac{2}{3}\pi$	$\cdots$	π
$f'(x)$		$-$	0	$+$	
$f(x)$	0	$\searrow$	$\frac{\sqrt{3}}{3}a$	$\nearrow$	0

When $a<0$ (※)



From the variation table, the maximum value is 0, at $x=0,\ \pi$.

Therefore, it is not applicable.

From (i)~(iii), $\boldsymbol{a=3}$

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Ex. Find the minimum value of area S of the triangle formed by the x-axis, the y-axis and the tangent to parabola $y=1-x^2$ at point $(t, 1-t^2)$. $(t>0)$

[Sol] Since $y=1-x^2$, $y'=-2x$

The equation of the tangent at point $(t, 1-t^2)$ is

$y-(1-t^2)=-2t(x-t)$ ⟵ The equation of the tangent at point $(a, f(a))$ is $y-f(a)=f'(a)(x-a)$

When $y=0$, $x=\dfrac{t^2+1}{2t}$. When $x=0$, $y=t^2+1$.

Therefore, let P and Q be the points of intersection of this tangent with the x-axis and the y-axis respectively.

$\mathrm{P}\left(\dfrac{t^2+1}{2t}, 0\right)$, $\mathrm{Q}(0, t^2+1)$

The y-coordinate on the x-axis is 0.
The x-coordinate on the y-axis is 0.

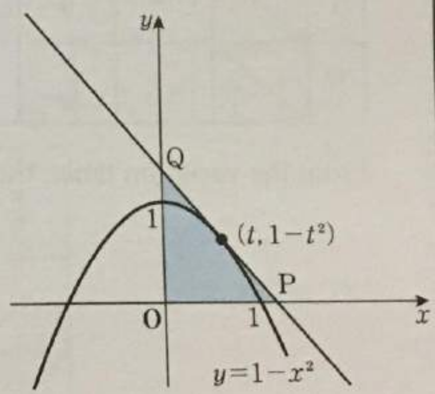
Thus,

$$S=\frac{1}{2}\cdot\frac{t^2+1}{2t}\cdot(t^2+1)=\frac{(t^2+1)^2}{4t}$$ ⟵ $S=\frac{1}{2}\mathrm{OP}\cdot\mathrm{OQ}$

$$S'=\frac{2(t^2+1)\cdot 2t\cdot 4t-(t^2+1)^2\cdot 4}{16t^2}=\frac{(t^2+1)(3t^2-\boxed{1})}{4t^2}$$

When $S'=0$ in $t>0$, $t=\boxed{\dfrac{\sqrt{3}}{3}}$

t	0	…	$\frac{\sqrt{3}}{3}$	…
S'		−	0	+
S		↘	$\frac{4\sqrt{3}}{9}$	↗



From the variation table,

the minimum value of S is $\boxed{\dfrac{4\sqrt{3}}{9}}$.

Answers: 1, $\frac{\sqrt{3}}{3}$,

t	0	…	$\frac{\sqrt{3}}{3}$	…
S'		−	0	+
S		↘	$\frac{4\sqrt{3}}{9}$	↗

, $\frac{4\sqrt{3}}{9}$

O36b

1. Find the maximum value of area S of the triangle formed by the x-axis, the y-axis and the tangent to curve $y=e^{-x}$ at point (t, e^{-t}). $(t>0)$

[Sol] Since $y=e^{-x}$, $y'=-e^{-x}$

The equation of the tangent at point (t, e^{-t}) is

$$y-e^{-t}=-e^{-t}(x-t)$$

When $y=0$, $x=t+1$. When $x=0$, $y=(t+1)e^{-t}$.

Therefore, let P and Q be the points of intersection of this tangent with the x-axis and the y-axis respectively.

$$\mathrm{P}(t+1, 0),\ \mathrm{Q}(0, (t+1)e^{-t})$$

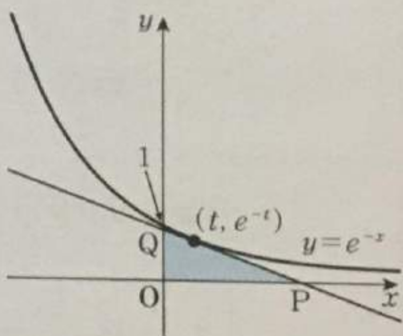
Thus,

$$S=\frac{1}{2}\cdot(t+1)\cdot(t+1)e^{-t}=\frac{1}{2}(t+1)^2e^{-t}$$

$$S'=\frac{1}{2}[2(t+1)e^{-t}-(t+1)^2e^{-t}]=-\frac{1}{2}(t+1)(t-1)e^{-t}$$

When $S'=0$ in $t>0$, $t=1$

t	0	$\cdots$	1	$\cdots$
S'	/	+	0	−
S	/	↗	$\frac{2}{e}$	↘



From the variation table, the maximum value of S is $\frac{2}{e}$.

Maxima and Minima

Name

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Ex. Given that an isosceles triangle ABC where AB=AC is inscribed in a circle with radius 1, find the maximum value of area S of $\triangle$ABC.

[Sol] Let $\angle BAC = 2\theta$. Since $0 < 2\theta < \pi$, $0 < \theta < \frac{\pi}{2}$

Let the center of the circle be O.

Dropping a perpendicular OM from point O to side AB,

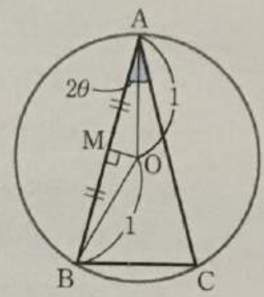
$AB = 2AM = 2OA\cos\theta = 2\cos\theta$ ← OA bisects $\angle$BAC.

Therefore,

$S = \frac{1}{2}AB \cdot AC\sin 2\theta$ ← $S = \frac{1}{2}bc\sin A$

$= \frac{1}{2}(2\cos\theta)^2 \sin 2\theta$

$= 4\cos^3\theta\sin\theta$ ← $\sin 2\alpha = 2\sin\alpha\cos\alpha$



$S' = 4[3\cos^2\theta \cdot (-\sin\theta) \cdot \sin\theta + \cos^3\theta \cdot \cos\theta]$

$= 4\cos^2\theta(-3\sin^2\theta + \cos^2\theta)$

$= 4\cos^2\theta(4\cos^2\theta - 3)$

$= 4\cos^2\theta(2\cos\theta + \boxed{\sqrt{3}})(2\cos\theta - \boxed{\sqrt{3}})$

When $S' = 0$ in $0 < \theta < \frac{\pi}{2}$, $\cos\theta = \boxed{\frac{\sqrt{3}}{2}}$, i.e. $\theta = \boxed{\frac{\pi}{6}}$

θ	0	…	$\frac{\pi}{6}$	…	$\frac{\pi}{2}$
S'		+	0	−	
S		↗	$\frac{3\sqrt{3}}{4}$	↘	

From the variation table, the maximum value of S is $\boxed{\frac{3\sqrt{3}}{4}}$.

Answers: $\sqrt{3}$, $\sqrt{3}$, $\frac{\sqrt{3}}{2}$, $\frac{\pi}{6}$,

θ	0	…	$\frac{\pi}{6}$	…	$\frac{\pi}{2}$
S'		+	0	−	
S		↗	$\frac{3\sqrt{3}}{4}$	↘	

, $\frac{3\sqrt{3}}{4}$

O37b

1. Given that an isosceles triangle ABC where AB=AC is inscribed in a circle with radius 1, find the maximum value of sum y of the lengths of the three sides of $\triangle$ABC.

[Sol] Let $\angle BAC = 2\theta$. Since $0 < 2\theta < \pi$, $0 < \theta < \frac{\pi}{2}$

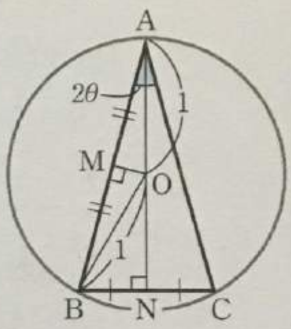
Let the center of the circle be O.

Dropping a perpendicular OM from point O to side AB,

$AB = 2AM = 2OA\cos\theta = 2\cos\theta$

Then, dropping a perpendicular AN from point A to side BC,

$BC = 2BN = 2AB\sin\theta = 4\cos\theta\sin\theta$



Therefore,

$y = 2AB + BC$ ← $\boxed{y = AB + AC + BC = 2AB + BC}$

$= 4(\cos\theta + \cos\theta\sin\theta)$

$= 4\cos\theta(1 + \sin\theta)$

$y' = 4[-\sin\theta(1+\sin\theta) + \cos\theta \cdot \cos\theta]$

$= -4(\sin\theta + \sin^2\theta - \cos^2\theta)$

$= -4(2\sin^2\theta + \sin\theta - 1)$

$= -4(\sin\theta + 1)(2\sin\theta - 1)$

When $y' = 0$ in $0 < \theta < \frac{\pi}{2}$, $\sin\theta = \frac{1}{2}$, i.e. $\theta = \frac{\pi}{6}$

θ	0	$\cdots$	$\frac{\pi}{6}$	$\cdots$	$\frac{\pi}{2}$
y'	/	+	0	−	/
y	/	↗	$3\sqrt{3}$	↘	/

From the variation table, the maximum value of y is $3\sqrt{3}$.

Maxima and Minima

Name

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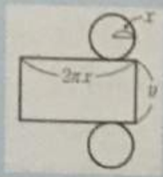
1. Given a right cylinder with volume 16π, find the ratio of the radius of the base to the height for which its surface area is a minimum.

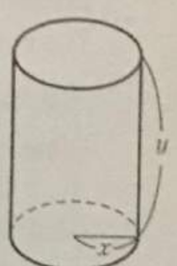
[Sol] Let x be the radius of the base, y be the height and S be the surface area.

$$16\pi=\pi x^2 y \quad \cdots①$$

$$S=2\pi x^2+2\pi xy \quad \cdots②$$

Volume = (base area) × (height)
Surface area = (base area) × 2 + (side area)





From ①, $y=\dfrac{16}{x^2}$ $\cdots③$

From ② and ③, $S=2\pi x^2+\dfrac{32\pi}{x}$

$$S'=4\pi x-\frac{32\pi}{x^2}=\frac{4\pi(x^3-8)}{x^2}=\frac{4\pi(x-2)(x^2+2x+4)}{x^2}$$

a^3-b^3
$=(a-b)(a^2+ab+b^2)$

When $S'=0$ in $x>0$, $x=2$

x	0	$\cdots$	2	$\cdots$
S'		$-$	0	$+$
S		↘	Relative minimum	↗

$S=24\pi$ at $x=2$

From the variation table, S is a minimum at $x=2$.

Then, from ③, $y=4$

Therefore, the ratio of the radius of the base to the height is **1 : 2.**

Here, the minimum value of S does not have to be found. In such a case, if the minimum value is the relative extreme value, it is possible to write "Relative minimum" in the variation table without finding the minimum value of S. Likewise, in the case that the maximum value does not have to be found, it is possible to write "Relative maximum."

O38b

2. On the xy-plane, consider curve $C: y=x-\ln x \ (0<x<1)$ and point P on curve C. Let the point of intersection of the tangent to C at point P with the y-axis be Q, and the point of intersection of the normal to C at point P with the y-axis be R. Given that point P moves along curve C, find the minimum value of the length of line segment QR.

[Sol] Since $y=x-\ln x$, $y'=1-\dfrac{1}{x}=\dfrac{x-1}{x}$

Let the coordinates of point P be $(t, t-\ln t)$. $(0<t<1)$

The equation of the tangent at P is

$$y-(t-\ln t)=\frac{t-1}{t}(x-t)$$

When $x=0$, $y=1-\ln t$

$\therefore$ Q$(0, 1-\ln t)$ …①

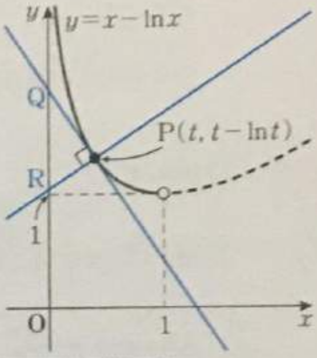
Also, the equation of the normal at P is

$$y-(t-\ln t)=-\frac{t}{t-1}(x-t)$$

When $x=0$, $y=\dfrac{2t^2-t}{t-1}-\ln t$

$\therefore$ R$\left(0, \dfrac{2t^2-t}{t-1}-\ln t\right)$ …②

From ① and ②, let the length of line segment QR be $f(t)$.



The equation of the normal at point $(a, f(a))$ is, when $f'(a)\neq 0$,
$y-f(a)=-\dfrac{1}{f'(a)}(x-a)$

$$f(t)=(1-\ln t)-\left(\frac{2t^2-t}{t-1}-\ln t\right)$$

$$=-\frac{2t^2-2t+1}{t-1}$$

(y-coordinate of Q)$-$(y-coordinate of R)

$$f'(t)=-\frac{(4t-2)(t-1)-(2t^2-2t+1)\cdot 1}{(t-1)^2}$$

$$=-\frac{2t^2-4t+1}{(t-1)^2}$$

When $f'(t)=0$ in $0<t<1$, $t=\dfrac{2-\sqrt{2}}{2}$

t	0	…	$\frac{2-\sqrt{2}}{2}$	…	1
$f'(t)$		$-$	0	$+$	
$f(t)$		↘	$2\sqrt{2}-2$	↗	

$2t^2-4t+1=0$
From Quadratic Formula II (J102)
$t=\dfrac{2\pm\sqrt{(-2)^2-2\cdot 1}}{2}$

From the variation table,
the minimum value of the length of line segment QR is $2\sqrt{2}-2$.

Maxima and Minima

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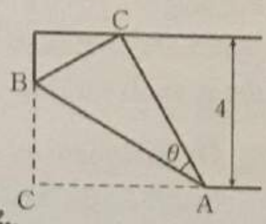
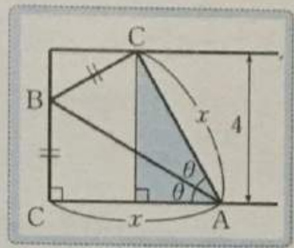
1. Folding a piece of paper of width 4 by placing its corner C on the opposite side, find θ for which the area of $\triangle ABC$ is a minimum. Then, find its area.



[Sol] Let $AC = x$, and let the area of $\triangle ABC$ be S.

Since $x \sin 2\theta = 4$ and $BC = x\tan\theta \left(0 < \theta < \frac{\pi}{4}\right)$,



$$S = \frac{1}{2} AC \cdot BC$$

$$= \frac{1}{2} x^2 \tan\theta$$

$$= \frac{1}{2} \cdot \left(\frac{4}{\sin 2\theta}\right)^2 \cdot \frac{\sin\theta}{\cos\theta}$$

$$= \frac{8\sin\theta}{\sin^2 2\theta \cos\theta}$$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$= \frac{2}{\sin\theta\cos^3\theta}$$

$$S' = -\frac{2[\cos\theta \cdot \cos^3\theta + \sin\theta \cdot 3\cos^2\theta \cdot (-\sin\theta)]}{\sin^2\theta\cos^6\theta}$$

$$= \frac{2(3\sin^2\theta - \cos^2\theta)}{\sin^2\theta\cos^4\theta}$$

$$= \frac{2(4\sin^2\theta - 1)}{\sin^2\theta\cos^4\theta}$$

$$= \frac{2(2\sin\theta + 1)(2\sin\theta - 1)}{\sin^2\theta\cos^4\theta}$$

When $S' = 0$ in $0 < \theta < \frac{\pi}{4}$, $\sin\theta = \frac{1}{2}$, i.e. $\theta = \frac{\pi}{6}$

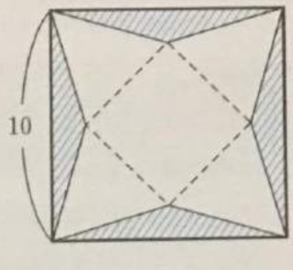
θ	0	$\cdots$	$\frac{\pi}{6}$	$\cdots$	$\frac{\pi}{4}$
S'	/	$-$	0	$+$	/
S	/	↘	$\frac{32\sqrt{3}}{9}$	↗	/

From the variation table,

the minimum value of the area of $\triangle ABC$ is $\frac{32\sqrt{3}}{9}$, at $\theta = \frac{\pi}{6}$.

O39b

2. Given a square of side 10cm, cut off isosceles triangles from each side (shaded area) and fold along the dotted line to make a square pyramid. Find the height (cm) for which the volume of this square pyramid is a maximum, then find the height (cm) of the isosceles triangles that are cut off.



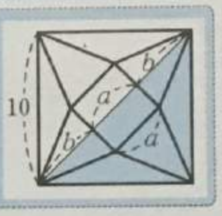
[Sol] Of the square pyramid, let the height be $x$cm, the volume be Vcm^3, the length of the side of the base be $a$cm and the height of the isosceles triangle faces be $b$cm.

$V=\frac{1}{3}a^2x$ $\cdots$① ← Volume $=\frac{1}{3}\times$(base area)$\times$(height)

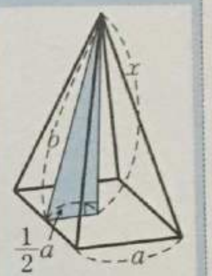
$a+2b=10\sqrt{2}$ $\cdots$② ← ※1

$b^2=\frac{1}{4}a^2+x^2$ $\cdots$③ ← ※2

※1



※2



From ②, $b=\dfrac{10\sqrt{2}-a}{2}$ $\cdots$④

From ③ and ④, $a=\dfrac{\sqrt{2}(50-x^2)}{10}$ $\cdots$⑤

From ① and ⑤, $V=\dfrac{(50-x^2)^2x}{150}$

$$V'=\frac{2(50-x^2)\cdot(-2x)\cdot x+(50-x^2)^2\cdot 1}{150}=\frac{(x^2-10)(x^2-50)}{30}$$

From ④, $0<b<5\sqrt{2}$; therefore, from ③, $0<x<5\sqrt{2}$

When $V'=0$ in $0<x<5\sqrt{2}$, $x=\sqrt{10}$

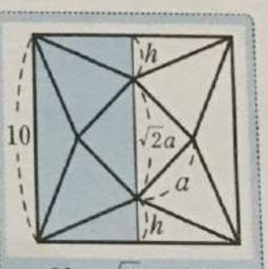
x	0	$\cdots$	$\sqrt{10}$	$\cdots$	$5\sqrt{2}$
V'		+	0	−	
V		↗	Relative maximum	↘	

$V=\dfrac{32\sqrt{10}}{3}$ at $x=\sqrt{10}$

From the variation table, V is a maximum at $x=\sqrt{10}$. From ⑤, $a=4\sqrt{2}$

Therefore, let $h$cm be the height of the isosceles triangles that are cut off.

$h=\frac{1}{2}(10-\sqrt{2}\cdot 4\sqrt{2})=1$ ←



$2h+\sqrt{2}a=10$

Ans. The height of the square pyramid: $\sqrt{10}$ cm

The height of the isosceles triangles: **1** cm

Maxima and Minima

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1. Find the maximum and minimum values of each given function.

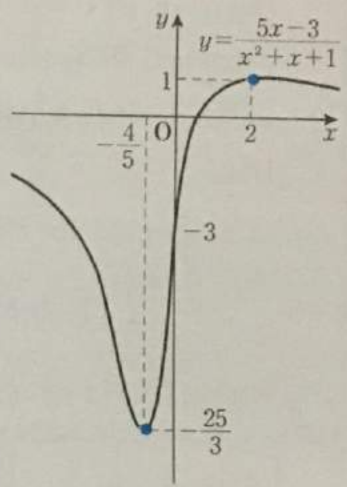
(1) $y=\dfrac{5x-3}{x^2+x+1}$

➡ O33

[Sol] $y'=\dfrac{5(x^2+x+1)-(5x-3)(2x+1)}{(x^2+x+1)^2}=-\dfrac{(5x+4)(x-2)}{(x^2+x+1)^2}$

When $y'=0$, $x=-\dfrac{4}{5}$, 2

x	$\cdots$	$-\dfrac{4}{5}$	$\cdots$	2	$\cdots$
y'	$-$	0	$+$	0	$-$
y	↘	$-\dfrac{25}{3}$	↗	1	↘



Also, since $\lim\limits_{x\to\infty} y=0$ and $\lim\limits_{x\to-\infty} y=0$,
from the variation table,

the maximum value is 1, at $x=2$ and
the minimum value is $-\dfrac{25}{3}$, at $x=-\dfrac{4}{5}$.

(2) $y=x\sqrt{2x-x^2}$

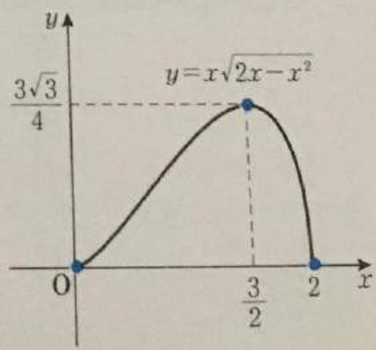
➡ O34

[Sol] Since $2x-x^2 \geqslant 0$, the domain is $0 \leqslant x \leqslant 2$.

$y'=1\cdot\sqrt{2x-x^2}+x\cdot\dfrac{2-2x}{2\sqrt{2x-x^2}}=\dfrac{x(3-2x)}{\sqrt{2x-x^2}}$

When $y'=0$ in $0<x<2$, $x=\dfrac{3}{2}$

x	0	$\cdots$	$\dfrac{3}{2}$	$\cdots$	2
y'		$+$	0	$-$	
y	0	↗	$\dfrac{3\sqrt{3}}{4}$	↘	0



From the variation table,

the maximum value is $\dfrac{3\sqrt{3}}{4}$, at $x=\dfrac{3}{2}$ and
the minimum value is 0, at $x=0,\ 2$.

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O40b

2. Find the maximum value of area S of the triangle formed by the x-axis, the y-axis and the tangent to curve $y=-\ln x$ at point $(t, -\ln t)$. $(0<t<1)$

➡ O36

[Sol] Since $y=-\ln x$, $y'=-\frac{1}{x}$

The equation of the tangent at point $(t, -\ln t)$ is

$$y-(-\ln t)=-\frac{1}{t}(x-t)$$

When $y=0$, $x=t(1-\ln t)$. When $x=0$, $y=1-\ln t$.

Therefore, let P and Q be the points of intersection of this tangent with the x-axis and the y-axis respectively.

$$\mathrm{P}(t(1-\ln t), 0),\ \mathrm{Q}(0, 1-\ln t)$$

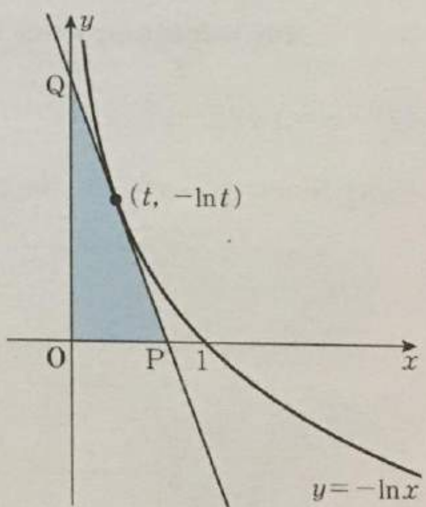
Thus,

$$S=\frac{1}{2}\cdot t(1-\ln t)\cdot(1-\ln t)=\frac{1}{2}t(1-\ln t)^2$$

$$S'=\frac{1}{2}\left[1\cdot(1-\ln t)^2+t\cdot 2(1-\ln t)\cdot\left(-\frac{1}{t}\right)\right]=-\frac{1}{2}(1-\ln t)(1+\ln t)$$

When $S'=0$ in $0<t<1$, $\ln t=-1$, i.e. $t=\frac{1}{e}$

t	0	$\cdots$	$\frac{1}{e}$	$\cdots$	1
S'		$+$	0	$-$	
S		$\nearrow$	$\frac{2}{e}$	$\searrow$	



From the variation table,

the maximum value of S is $\frac{2}{e}$.

Various Applications of Differentiation

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Using the graph, find the number of real solutions of each given equation.

Ex.

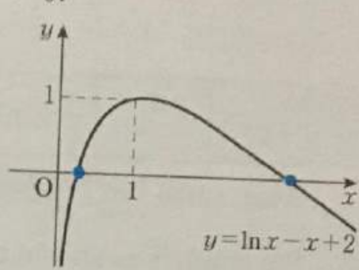
$\ln x - x + 2 = 0 \quad \left(\lim_{x\to\infty}(\ln x - x + 2) = -\infty\right)$

[Sol] Let $f(x) = \ln x - x + 2$. The domain is $x > 0$.

$$f'(x) = \frac{1}{x} - 1 = \frac{1-x}{x}$$

When $f'(x) = 0$ in $x > 0$, $x = 1$

x	0	$\cdots$	1	$\cdots$
$f'(x)$		$+$	0	$-$
$f(x)$		$\nearrow$	1	$\searrow$



Also, since $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = -\infty$, the graph is as shown on the right.

Therefore, the equation has 2 real solutions.

The real solutions of $f(x) = 0$ are the x-coordinates of the common points of function $y = f(x)$ and the x-axis.

(1) $\sqrt{x} + \frac{4}{x} - 4 = 0$

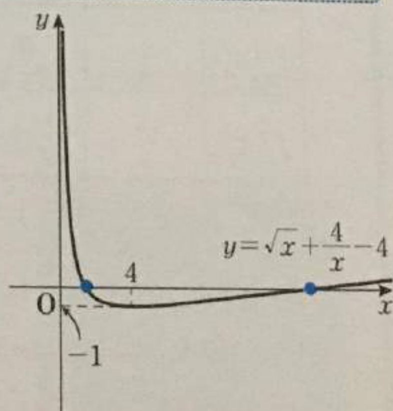
[Sol] Let $f(x) = \sqrt{x} + \frac{4}{x} - 4$. The domain is $x > 0$.

The expression in the square root $\geqq 0$ and also the denominator $\neq 0$

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{4}{x^2} = \frac{x\sqrt{x} - 8}{2x^2}$$

When $f'(x) = 0$ in $x > 0$, $x\sqrt{x} = 8$, i.e. $x = 4$

x	0	$\cdots$	4	$\cdots$
$f'(x)$		$-$	0	$+$
$f(x)$		$\searrow$	-1	$\nearrow$



Also, since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to 0^+} f(x) = \infty$, the graph is as shown on the right.

Therefore, the equation has **2 real solutions**.

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041b

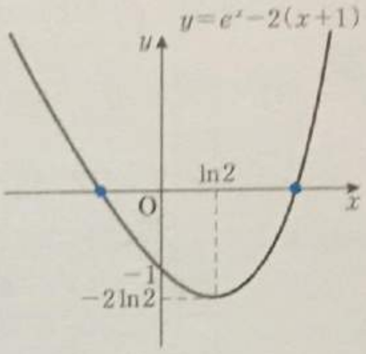
(2) $e^x-2(x+1)=0 \quad \left(\lim_{x\to\infty}[e^x-2(x+1)]=\infty\right)$

[Sol] Let $f(x)=e^x-2(x+1)$.

$f'(x)=e^x-2$

When $f'(x)=0$, $e^x=2$, i.e. $x=\ln 2$

x	$\cdots$	$\ln 2$	$\cdots$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	$-2\ln 2$	$\nearrow$



Also, since $\lim_{x\to\infty} f(x)=\infty$ and $\lim_{x\to-\infty} f(x)=\infty$,

the graph is as shown on the right.

Therefore, the equation has 2 **real solutions**.

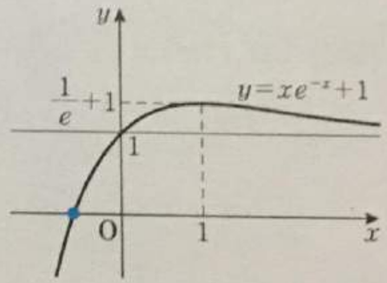
(3) $xe^{-x}=-1 \quad \left(\lim_{x\to\infty} xe^{-x}=0\right)$

[Sol] Let $f(x)=xe^{-x}+1$.

$f'(x)=1\cdot e^{-x}-xe^{-x}=(1-x)e^{-x}$

When $f'(x)=0$, $x=1$

x	$\cdots$	1	$\cdots$
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$	$\frac{1}{e}+1$	$\searrow$



Also, since $\lim_{x\to\infty} f(x)=1$ and $\lim_{x\to-\infty} f(x)=-\infty$,

the graph is as shown on the right.

Therefore, the equation has 1 **real solution**.

Since $\lim_{x\to\infty} xe^{-x}=0$,

$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}(xe^{-x}+1)=1$

O42a

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Ex. Given that k is a constant, find the number of real solutions of equation $x^3+2=kx$.

[Sol] Since $x=0$ is not a solution of the equation, $x \neq 0$ ⟵ When $x=0$, the equation does not hold true.

Dividing both sides of the equation by x, $\dfrac{x^3+2}{x}=k$

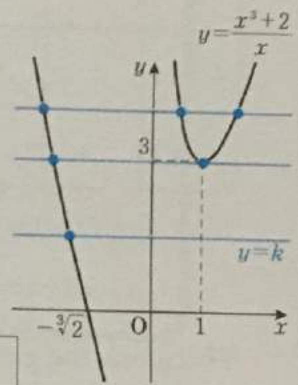
Let $f(x)=\dfrac{x^3+2}{x}$.

a^3-b^3
$=(a-b)(a^2+ab+b^2)$

$$f'(x)=\frac{3x^2\cdot x-(x^3+2)\cdot 1}{x^2}=\frac{2(x-1)(x^2+x+1)}{x^2}$$

When $f'(x)=0$, $x=1$

x	$\cdots$	0	$\cdots$	1	$\cdots$
$f'(x)$	$-$	╱	$-$	0	$+$
$f(x)$	↘	╱	↘	3	↗



Also, $\lim\limits_{x\to\infty} f(x)=\boxed{\infty}$, $\lim\limits_{x\to-\infty} f(x)=\boxed{\infty}$

$\lim\limits_{x\to 0^+} f(x)=\boxed{\infty}$, $\lim\limits_{x\to 0^-} f(x)=\boxed{-\infty}$

Since $f(x)=x^2+\dfrac{2}{x}$

Therefore, the graph of $y=f(x)$ is as shown on the right.

From the common points of this graph and line $y=k$, the number of real solutions is:

When $k>3$, $\boxed{3}$ real solution(s)

When $k=3$, $\boxed{2}$ real solution(s) ⟵

When $k<3$, $\boxed{1}$ real solution(s)

The number of common point(s) of the graphs of $y=\dfrac{x^3+2}{x}$ and $y=k$ is equal to the number of real solution(s).

Answers: ∞, ∞, ∞, −∞, 3, 2, 1

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O42b

1. Given that k is a constant, find the number of real solutions of equation $e^x = kx$.

$\left(\lim_{x\to\infty}\frac{e^x}{x}=\infty\right)$

[Sol] Since $x=0$ is not a solution of the equation, $x\neq 0$

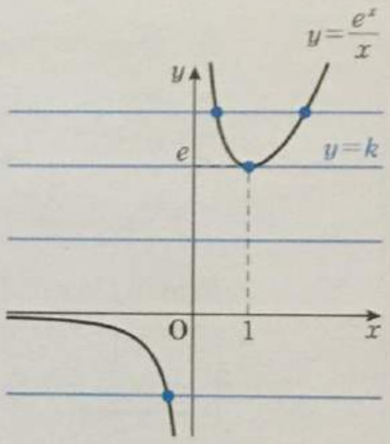
Dividing both sides of the equation by x, $\frac{e^x}{x}=k$

Let $f(x)=\frac{e^x}{x}$.

$$f'(x)=\frac{e^x\cdot x-e^x\cdot 1}{x^2}=\frac{(x-1)e^x}{x^2}$$

When $f'(x)=0$, $x=1$

x	$\cdots$	0	$\cdots$	1	$\cdots$
$f'(x)$	$-$		$-$	0	$+$
$f(x)$	↘		↘	e	↗



Also, $\lim_{x\to\infty}f(x)=\infty$, $\lim_{x\to-\infty}f(x)=0$

$\lim_{x\to 0^+}f(x)=\infty$, $\lim_{x\to 0^-}f(x)=-\infty$

Therefore, the graph of $y=f(x)$ is as shown above.

From the common points of this graph and line $y=k$, the number of real solutions is:

When $k>e$, 2 real solutions

When $k<0$, $k=e$, 1 real solution

When $0\leqslant k<e$, no real solutions

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Prove each given inequality and determine the value of x for which the equation holds true.

Ex. $e^x \geqq 1+x$

[Sol] Let $f(x)=e^x-(1+x)$.

$f'(x)=e^x-1$

When $f'(x)=0$, $e^x=1$, i.e. $x=0$

x	$\cdots$	0	$\cdots$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	0	$\nearrow$

Therefore, $f(x)$ has a minimum value of 0, at $x=0$.

$\therefore$ $f(x) \geqq 0$, i.e. $e^x \geqq 1+x$

When $x=0$, the equation holds true.

(1) $xe^x-e^x \geqq -1$

[Sol] Let $f(x)=xe^x-e^x+1$.

$f'(x)=(1\cdot e^x+xe^x)-e^x=xe^x$

When $f'(x)=0$, $x=0$

x	$\cdots$	0	$\cdots$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	0	$\nearrow$

Therefore, $f(x)$ has a minimum value of 0, at $x=0$.

$\therefore$ $f(x) \geqq 0$, i.e. $xe^x-e^x \geqq -1$

When $x=0$, the equation holds true.

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(2) $x\ln x \geqslant x-1$

[Sol] Let $f(x)=x\ln x-(x-1)$. The domain is $x>0$.

$$f'(x)=\left(1\cdot\ln x+x\cdot\frac{1}{x}\right)-1=\ln x$$

When $f'(x)=0$ in $x>0$, $\ln x=0$, i.e. $x=1$

x	0	$\cdots$	1	$\cdots$
$f'(x)$		$-$	0	$+$
$f(x)$		$\searrow$	0	$\nearrow$

Therefore, $f(x)$ has a minimum value of 0, at $x=1$.

$\therefore\ f(x)\geqslant 0$

Thus, when $x>0$, $x\ln x\geqslant x-1$

When $\boldsymbol{x=1}$, the equation holds true.

(3) $\sin x \geqslant x\cos x \quad (0\leqslant x\leqslant \pi)$

[Sol] Let $f(x)=\sin x-x\cos x$.

$$f'(x)=\cos x-[1\cdot\cos x+x\cdot(-\sin x)]=x\sin x$$

In $0<x<\pi$, there is no x that satisfies $f'(x)=0$.

x	0	$\cdots$	π
$f'(x)$		$+$	
$f(x)$	0	$\nearrow$	π

Therefore, $f(x)$ has a minimum value of 0, at $x=0$.

$\therefore\ f(x)\geqslant 0$

Thus, when $0\leqslant x\leqslant\pi$, $\sin x\geqslant x\cos x$

When $\boldsymbol{x=0}$, the equation holds true.

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Prove each given inequality.

Ex.

$$e^x > 1 + x + \frac{x^2}{2} \quad (x > 0)$$

[Sol] Let $f(x) = e^x - \left(1 + x + \frac{x^2}{2}\right)$.

$$f'(x) = e^x - 1 - x$$
$$f''(x) = e^x - 1$$

Since $f'(x) > 0$ cannot be confirmed with this expression, determine $f''(x)$.

When $x > 0$, $e^x > 1$; therefore, $f''(x) > 0$

Thus, since $f'(x)$ increases in $x \geqslant 0$

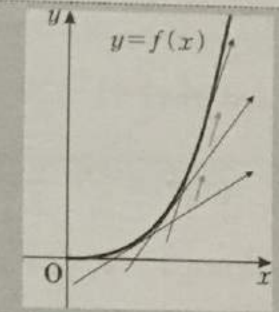
and $f'(0) = 0$, $f'(x) > 0$

Therefore, since $f(x)$ increases in $x \geqslant 0$

and $f(0) = 0$, $f(x) > 0$

Thus, when $x > 0$,

$$e^x > 1 + x + \frac{x^2}{2}$$



If $f''(x) > 0$, the curve $y = f(x)$ is concave up and $f'(x)$ increases.

(1) $x\sin x < 2(1 - \cos x) \quad \left(0 < x \leqslant \frac{\pi}{2}\right)$

[Sol] Let $f(x) = 2(1 - \cos x) - x\sin x$.

$$f'(x) = 2\sin x - (1 \cdot \sin x + x\cos x) = \sin x - x\cos x$$
$$f''(x) = \cos x - [1 \cdot \cos x + x \cdot (-\sin x)] = x\sin x$$

When $0 < x \leqslant \frac{\pi}{2}$, $\sin x > 0$; therefore, $f''(x) > 0$

Thus, since $f'(x)$ increases in $0 \leqslant x \leqslant \frac{\pi}{2}$ and $f'(0) = 0$, $f'(x) > 0$

Therefore, since $f(x)$ increases in $0 \leqslant x \leqslant \frac{\pi}{2}$ and $f(0) = 0$, $f(x) > 0$

Thus, when $0 < x \leqslant \frac{\pi}{2}$, $x\sin x < 2(1 - \cos x)$

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O44b

(2) $1+\frac{1}{2}x-\frac{1}{8}x^2 < \sqrt{1+x} < 1+\frac{1}{2}x \quad (x>0)$

[Sol] Let $f(x)=\sqrt{1+x}-\left(1+\frac{1}{2}x-\frac{1}{8}x^2\right)$.

$$f'(x)=\frac{1}{2\sqrt{1+x}}-\frac{1}{2}+\frac{1}{4}x$$

$$f''(x)=-\frac{1}{4(\sqrt{1+x})^3}+\frac{1}{4}=\frac{(\sqrt{1+x})^3-1}{4(\sqrt{1+x})^3}$$

When $x>0$, $(\sqrt{1+x})^3>1$; therefore, $f''(x)>0$

Thus, since $f'(x)$ increases in $x \geqslant 0$ and $f'(0)=0$, $f'(x)>0$

Therefore, since $f(x)$ increases in $x \geqslant 0$ and $f(0)=0$, $f(x)>0$

Thus, when $x>0$, $1+\frac{1}{2}x-\frac{1}{8}x^2 < \sqrt{1+x}$ ···①

Let $g(x)=1+\frac{1}{2}x-\sqrt{1+x}$.

$$g'(x)=\frac{1}{2}-\frac{1}{2\sqrt{1+x}}=\frac{\sqrt{1+x}-1}{2\sqrt{1+x}}$$

When $x>0$, $\sqrt{1+x}>1$; therefore, $g'(x)>0$

Thus, since $g(x)$ increases in $x \geqslant 0$ and $g(0)=0$, $g(x)>0$

Therefore, when $x>0$, $\sqrt{1+x} < 1+\frac{1}{2}x$ ···②

From ① and ②, when $x>0$, $1+\frac{1}{2}x-\frac{1}{8}x^2 < \sqrt{1+x} < 1+\frac{1}{2}x$

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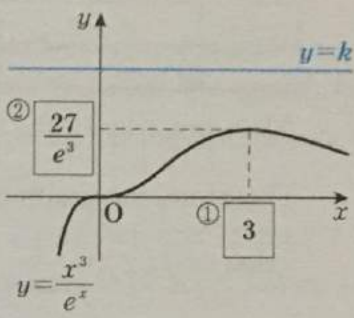
Ex. Find the range of constant k such that inequality $x^3 \leqslant ke^x$ is satisfied for all real numbers x. $\left(\lim_{x\to\infty}\frac{x^3}{e^x}=0\right)$

[Sol] Since $e^x>0$, $\frac{x^3}{e^x}\leqslant k$

Let $f(x)=\frac{x^3}{e^x}$. $f'(x)=\frac{3x^2e^x-x^3e^x}{e^{2x}}=\frac{x^2(\boxed{3-x})}{e^x}$

When $f'(x)=0$, $x=0$, $\boxed{3}$

x	$\cdots$	0	$\cdots$	3	$\cdots$
$f'(x)$	$+$	0	$+$	0	$-$
$f(x)$	$\nearrow$	0	$\nearrow$	$\frac{27}{e^3}$	$\searrow$



Also, $\lim_{x\to\infty}f(x)=0$, $\lim_{x\to-\infty}f(x)=-\infty$

Therefore, the graph of $y=f(x)$ is as shown above, and the maximum value of $f(x)$ is $\boxed{\frac{27}{e^3}}$.

Thus, if $k\geqslant\boxed{\frac{27}{e^3}}$, then $\frac{x^3}{e^x}\leqslant k$, i.e. $x^3\leqslant ke^x$ is satisfied for all real numbers x.

$\therefore\ k\geqslant\boxed{\frac{27}{e^3}}$

Answers: $3-x$, 3,

x	$\cdots$	0	$\cdots$	3	$\cdots$
$f'(x)$	$+$	0	$+$	0	$-$
$f(x)$	$\nearrow$	0	$\nearrow$	$\frac{27}{e^3}$	$\searrow$

, $\frac{27}{e^3}$, $\frac{27}{e^3}$, $\frac{27}{e^3}$,

(on the graph) ① 3, ② $\frac{27}{e^3}$

O45b

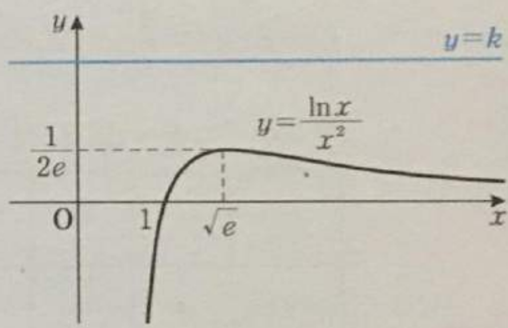
1. Find the minimum value of constant k such that inequality $kx^2 \geqslant \ln x$ is satisfied for all positive numbers x. $\left(\lim_{x\to\infty}\frac{\ln x}{x^2}=0\right)$

[Sol] Since $x>0$, $k \geqslant \frac{\ln x}{x^2}$

Let $f(x)=\frac{\ln x}{x^2}$. $f'(x)=\frac{\frac{1}{x}\cdot x^2-\ln x\cdot 2x}{x^4}=\frac{1-2\ln x}{x^3}$

When $f'(x)=0$ in $x>0$, $\ln x=\frac{1}{2}$, i.e. $x=\sqrt{e}$

x	0	$\cdots$	$\sqrt{e}$	$\cdots$
$f'(x)$		$+$	0	$-$
$f(x)$		$\nearrow$	$\frac{1}{2e}$	$\searrow$



Also, $\lim_{x\to\infty}f(x)=0$, $\lim_{x\to 0^+}f(x)=-\infty$

Therefore, the graph of $y=f(x)$ is as shown above, and the maximum value of $f(x)$ is $\frac{1}{2e}$.

Thus, if $k \geqslant \frac{1}{2e}$, then $k \geqslant \frac{\ln x}{x^2}$, i.e. $kx^2 \geqslant \ln x$ is satisfied for all positive numbers x.

Therefore, the minimum value of k is $\frac{1}{2e}$.

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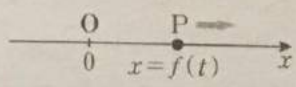
Let $x=f(t)$ be the coordinate of point P moving on a number line at time t. The rate of change of x at time t is called the ***velocity***, and the rate of change of velocity is called the ***acceleration***.



Therefore, the following statement is true.

Velocity and Acceleration

Let $x=f(t)$ be the coordinate of point P moving on a number line at time t. Velocity v and acceleration α of point P at time t can be determined as follows.

$$v=\frac{dx}{dt}=f'(t),\quad \alpha=\frac{dv}{dt}=\frac{d^2x}{dt^2}=f''(t)$$

Also, the absolute value of velocity v (written as $|v|$) is called the ***speed***, and the absolute value of acceleration α (written as $|\alpha|$) is called the ***magnitude of acceleration***.

1. Given that the coordinate of point P moving on a number line at time t is expressed as follows, find the speed $|v|$ and the magnitude of acceleration $|\alpha|$ of point P at the time shown in the brackets [ ].

Ex.

$x=2\sin\pi t \quad \left[t=\frac{1}{4}\right]$

[Sol] $\frac{dx}{dt}=2\pi\cos\pi t,\ \frac{d^2x}{dt^2}=-2\pi^2\sin\pi t$

$\therefore\ |v|=\left|2\pi\cos\left(\pi\cdot\frac{1}{4}\right)\right|=\sqrt{2}\pi,\ |\alpha|=\left|-2\pi^2\sin\left(\pi\cdot\frac{1}{4}\right)\right|=\sqrt{2}\pi^2$

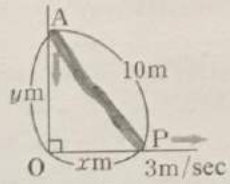
(1) $x=6\sin\frac{\pi}{6}t \quad [t=4]$

[Sol] $\frac{dx}{dt}=\pi\cos\frac{\pi}{6}t,\ \frac{d^2x}{dt^2}=-\frac{\pi^2}{6}\sin\frac{\pi}{6}t$

$\therefore\ |v|=\left|\pi\cos\left(\frac{\pi}{6}\cdot 4\right)\right|=\frac{\pi}{2},\ |\alpha|=\left|-\frac{\pi^2}{6}\sin\left(\frac{\pi}{6}\cdot 4\right)\right|=\frac{\sqrt{3}}{12}\pi^2$

O46b

Ex. There is a 10m long pole leaning against a vertical wall. Given that the bottom of this pole, P, moves away from the wall along the ground at a rate of 3m/sec, find the speed at which the top of this pole, A, is sliding down the wall when P is 6m away from the wall.



[Sol] Let AO be the perpendicular from point A along the wall to the ground. Let xm and ym be OP and OA after t seconds respectively.

$x^2+y^2=100$ ⋯①

Since x and y are functions of t, differentiating x^2 and y^2 with respect to t, $2x\cdot\dfrac{dx}{dt}$, $2y\cdot\dfrac{dy}{dt}$

Differentiating both sides of ① with respect to t,

$$2x\cdot\frac{dx}{dt}+2y\cdot\frac{dy}{dt}=0 \qquad \therefore\ \frac{dy}{dt}=-\frac{x}{y}\cdot\frac{dx}{dt} \cdots②$$

Then, when $x=6$, from ①, $y=8$ ← $y^2=100-x^2$, $y>0$

Also, since the pole moves away from the wall at a rate of 3m/sec,

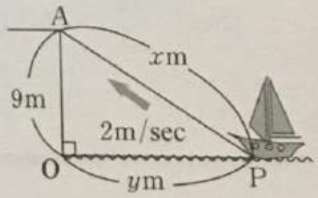
$$\left|\frac{dx}{dt}\right|=3$$

Therefore, from ②, the speed at which A is sliding down is

$$\left|\frac{dy}{dt}\right|=\left|-\frac{x}{y}\cdot\frac{dx}{dt}\right|=\frac{6}{8}\cdot 3=\frac{9}{4}$$

Ans. $\frac{9}{4}$ m/sec

2. Boat P is pulled by a rope from the top of the pier, point A, which is 9m above sea level. If the boat is pulled at a rate of 2m/sec, find the speed of boat P when the length of the rope from point A to the boat is 15m.



[Sol] Let AO be the perpendicular from point A down the side of the pier to the water surface. Let xm and ym be AP and OP after t seconds respectively.

$x^2=y^2+81$ ⋯①

Differentiating both sides of ① with respect to t,

$$2x\cdot\frac{dx}{dt}=2y\cdot\frac{dy}{dt} \qquad \therefore\ \frac{dy}{dt}=\frac{x}{y}\cdot\frac{dx}{dt} \cdots②$$

Then, when $x=15$, from ①, $y=12$

Also, since the rope is pulled at a rate of 2m/sec, $\left|\dfrac{dx}{dt}\right|=2$

Therefore, from ②, the speed of P pulled to the pier is

$$\left|\frac{dy}{dt}\right|=\left|\frac{x}{y}\cdot\frac{dx}{dt}\right|=\frac{15}{12}\cdot 2=\frac{5}{2}$$

Ans. $\frac{5}{2}$ m/sec

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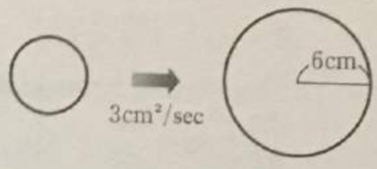
Ex. The area of a circle increases at a rate of $3\text{cm}^2/\text{sec}$. Find the speed at which the radius of this circle is increasing when the radius is 6cm.



[Sol] Let $r$cm and $S\text{cm}^2$ be the radius and the area of this circle after t seconds respectively.

$S=\pi r^2$

Differentiating both sides with respect to t,

$\dfrac{dS}{dt}=2\pi r\cdot\dfrac{dr}{dt}$ …①

Since r is a function of t, differentiating r^2 with respect to t, $2r\cdot\dfrac{dr}{dt}$

Since the area increases at a rate of $3\text{cm}^2/\text{sec}$,

$\dfrac{dS}{dt}=3$ …②

From ① and ②, $\dfrac{dr}{dt}=\dfrac{3}{2\pi r}$ $\boxed{3=2\pi r\cdot\dfrac{dr}{dt}}$

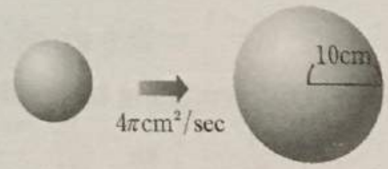
Therefore, when $r=6$, $\dfrac{dr}{dt}=\dfrac{3}{2\pi\cdot 6}=\dfrac{1}{4\pi}$

Ans. $\dfrac{1}{4\pi}$ cm/sec

1. The surface area of a sphere increases at a rate of $4\pi\text{cm}^2/\text{sec}$. Find the speed at which the radius of this sphere is increasing when the radius is 10cm.

[Sol] Let $r$cm and $S\text{cm}^2$ be the radius and the surface area of this sphere after t seconds respectively.

$S=4\pi r^2$



Differentiating both sides with respect to t,

$\dfrac{dS}{dt}=8\pi r\cdot\dfrac{dr}{dt}$ …①

Since the surface area increases at a rate of $4\pi\text{cm}^2/\text{sec}$,

$\dfrac{dS}{dt}=4\pi$ …②

From ① and ②, $\dfrac{dr}{dt}=\dfrac{1}{2r}$

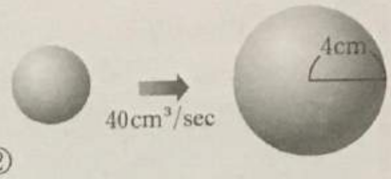
Therefore, when $r=10$, $\dfrac{dr}{dt}=\dfrac{1}{2\cdot 10}=\dfrac{1}{20}$ Ans. $\dfrac{1}{20}$ cm/sec

The surface area of a sphere with radius r is $4\pi r^2$.

2. The volume of a bubble increases at a rate of $40\text{cm}^3/\text{sec}$. Find the speed at which the surface area of this bubble is increasing when the radius is 4cm.

[Sol] Let $r$cm, $S\text{cm}^2$ and $V\text{cm}^3$ be the radius, surface area and volume of this bubble after t seconds respectively.

$$S=4\pi r^2,\ V=\frac{4}{3}\pi r^3$$



Differentiating both sides with respect to t,

$$\frac{dS}{dt}=8\pi r\cdot\frac{dr}{dt}\ \cdots①,\quad \frac{dV}{dt}=4\pi r^2\cdot\frac{dr}{dt}\ \cdots②$$

Since the volume increases at a rate of $40\text{cm}^3/\text{sec}$,

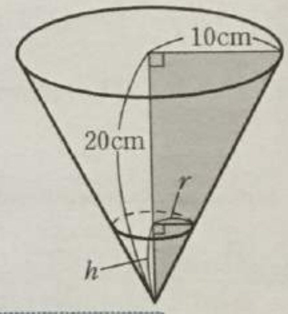
$$\frac{dV}{dt}=40\ \cdots③$$

From ② and ③, $\dfrac{dr}{dt}=\dfrac{10}{\pi r^2}\ \cdots④$

From ① and ④, $\dfrac{dS}{dt}=8\pi r\cdot\dfrac{10}{\pi r^2}=\dfrac{80}{r}$

Therefore, when $r=4$, $\dfrac{dS}{dt}=\dfrac{80}{4}=20$ Ans. **20 cm^2/sec**

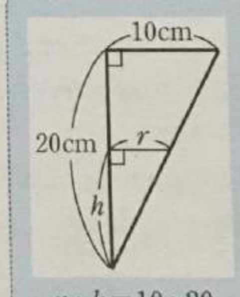
3. Water is poured at a rate of $3\text{cm}^3/\text{sec}$ into a cone-shaped container which has a base radius of 10cm at its widest point and a depth of 20cm. Find the speed at which the height of water is rising when the water is 6cm deep.



[Sol] Let $r$cm, $h$cm and $V\text{cm}^3$ be the radius of the water surface, the height of water and the amount of water after t seconds respectively.

Since $\dfrac{r}{h}=\dfrac{10}{20}$, $r=\dfrac{1}{2}h$; therefore,

$$V=\frac{1}{3}\pi r^2 h=\frac{1}{12}\pi h^3 \quad ※$$



$r : h=10 : 20$

Differentiating both sides with respect to t,

$$\frac{dV}{dt}=\frac{1}{4}\pi h^2\cdot\frac{dh}{dt}\ \cdots①$$

Since the water is poured at a rate of $3\text{cm}^3/\text{sec}$,

$$\frac{dV}{dt}=3\ \cdots②$$

※ To find the speed at which the height of water is rising, express the amount in terms of h (the height of water).

From ① and ②, $\dfrac{dh}{dt}=\dfrac{12}{\pi h^2}$

Therefore, when $h=6$, $\dfrac{dh}{dt}=\dfrac{12}{\pi\cdot 6^2}=\dfrac{1}{3\pi}$ Ans. **$\dfrac{1}{3\pi}$ cm/sec**

The volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.

Various Applications of Differentiation

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When the function $f(x)$ is differentiable at $x=a$, $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

Therefore, when the value of h is approaching 0, $f'(a)\approx\frac{f(a+h)-f(a)}{h}$

Thus, the following statement is true.

Linear Approximation I

When the function $f(x)$ is differentiable at $x=a$ and the value of $|h|$ is approaching 0, $f(a+h)\approx f(a)+f'(a)h$

1. Using Linear Approximation I, express the approximate values of the following numbers in the decimal form.

Ex. $\ln 1.01$

[Sol] Let $f(x)=\ln x$. $f'(x)=\frac{1}{x}$

$f(a+h)\approx f(a)+f'(a)h$

$\therefore\ \ln 1.01=\ln(1+0.01)\approx \ln 1+\frac{1}{1}\cdot 0.01=0.01$

(1) $\sqrt{100.5}$

[Sol] Let $f(x)=\sqrt{x}$. $f'(x)=\frac{1}{2\sqrt{x}}$

$\therefore\ \sqrt{100.5}=\sqrt{100+0.5}\approx\sqrt{100}+\frac{1}{2\sqrt{100}}\cdot 0.5=\mathbf{10.025}$

(2) $\tan 46^\circ$ $\left(\text{Use } \frac{\pi}{90}=0.035\right)$

[Sol] Let $f(x)=\tan x$. $f'(x)=\boxed{\frac{1}{\cos^2 x}}$

$\therefore\ \tan 46^\circ=\tan\left(\frac{\pi}{4}+\boxed{\frac{\pi}{180}}\right)\approx\tan\frac{\pi}{4}+\frac{1}{\cos^2\frac{\pi}{4}}\cdot\frac{\pi}{180}$

$180^\circ=\pi$

$=1+\frac{\pi}{90}=\mathbf{1.035}$

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In Linear Approximation I on side a, if $a=0$ and $h=x$, then the following statement is true.

Linear Approximation II

When the value of $|x|$ is approaching 0, $f(x) \approx f(0)+f'(0)x$

2. When the value of $|x|$ is approaching 0, write the linear approximation of each given function.

Ex. $f(x)=\ln(1+x)$

[Sol] Since $f'(x)=\frac{1}{1+x}$,

$$f(x) \approx \ln(1+0)+\frac{1}{1+0} \cdot x = x \quad \Leftarrow \quad \boxed{f(x) \approx f(0)+f'(0)x}$$

(1) $f(x)=e^x$

[Sol] Since $f'(x)=e^x$,

$$f(x) \approx e^0+e^0 \cdot x = 1+x$$

(2) $f(x)=\tan x$

[Sol] Since $f'(x)=\frac{1}{\cos^2 x}$,

$$f(x) \approx \tan 0+\frac{1}{\cos^2 0} \cdot x = x$$

(3) $f(x)=(1+x)^p$

[Sol] Since $f'(x)=p(1+x)^{p-1}$,

$$f(x) \approx (1+0)^p+p(1+0)^{p-1} \cdot x = 1+px$$

Various Applications of Differentiation

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1. Find the range of values of p for which equation $\ln x = -x^2+3x+p$ has three different real solutions.

[Sol] Rearranging the equation, $\ln x + x^2 - 3x = p$

Let $f(x) = \ln x + x^2 - 3x$. The domain is $x > 0$.

$$f'(x) = \frac{1}{x} + 2x - 3 = \frac{(2x-1)(x-1)}{x}$$

When $f'(x)=0$ in $x>0$, $x=\frac{1}{2}$, 1

x	0	$\cdots$	$\frac{1}{2}$	$\cdots$	1	$\cdots$
$f'(x)$		$+$	0	$-$	0	$+$
$f(x)$		$\nearrow$	$-\ln 2 - \frac{5}{4}$	$\searrow$	-2	$\nearrow$

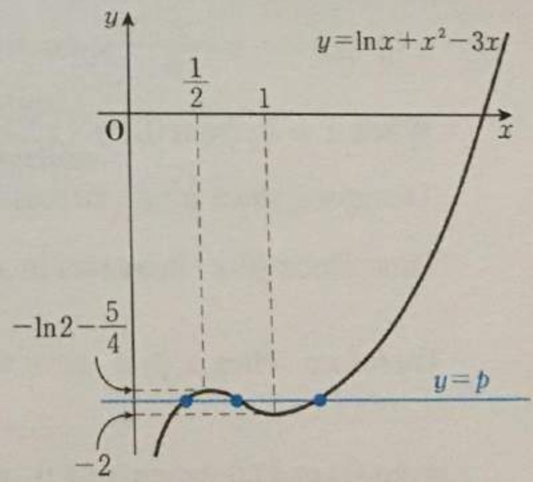
Also, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to 0^+} f(x) = -\infty$

Therefore, the graph of $y=f(x)$ is as shown below.

Then, find the range of p for which this graph and line $y=p$ have three common points.

Thus, the range of p is

$$-2 < p < -\ln 2 - \frac{5}{4}$$



O49b

2. Show that $x-\frac{x^3}{6} \leqslant \sin x \leqslant x-\frac{x^3}{6}+\frac{x^5}{120}$ holds true for all $x \geqslant 0$.

[Sol] Let $f(x)=\sin x-\left(x-\frac{x^3}{6}\right)$.

$f'(x)=\cos x-1+\frac{x^2}{2}$

$f''(x)=-\sin x+x$

$f'''(x)=-\cos x+1$

Since $f''(x) \geqslant 0$ cannot be confirmed with this expression, determine $f'''(x)$.

When $x \geqslant 0$, $-1 \leqslant \cos x \leqslant 1$; therefore, $f'''(x) \geqslant 0$

Thus, since $f''(x)$ increases in $x \geqslant 0$ and $f''(0)=0$, $f''(x) \geqslant 0$

Therefore, since $f'(x)$ increases in $x \geqslant 0$ and $f'(0)=0$, $f'(x) \geqslant 0$

Thus, since $f(x)$ increases in $x \geqslant 0$ and $f(0)=0$, $f(x) \geqslant 0$

Therefore, when $x \geqslant 0$, $x-\frac{x^3}{6} \leqslant \sin x$ $\cdots$①

Let $g(x)=x-\frac{x^3}{6}+\frac{x^5}{120}-\sin x$.

$g'(x)=1-\frac{x^2}{2}+\frac{x^4}{24}-\cos x$

$g''(x)=-x+\frac{x^3}{6}+\sin x$

From ①, $\sin x-\left(x-\frac{x^3}{6}\right) \geqslant 0$

When $x \geqslant 0$, from ①, $g''(x) \geqslant 0$

Therefore, since $g'(x)$ increases in $x \geqslant 0$ and $g'(0)=0$, $g'(x) \geqslant 0$

Thus, since $g(x)$ increases in $x \geqslant 0$ and $g(0)=0$, $g(x) \geqslant 0$

Therefore, when $x \geqslant 0$, $\sin x \leqslant x-\frac{x^3}{6}+\frac{x^5}{120}$ $\cdots$②

From ① and ②, when $x \geqslant 0$, $x-\frac{x^3}{6} \leqslant \sin x \leqslant x-\frac{x^3}{6}+\frac{x^5}{120}$

Various Applications of Differentiation

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1. Given that k is a constant, find the number of real solutions of equation $2\sqrt{x}-x+k=0$.

➡ O42

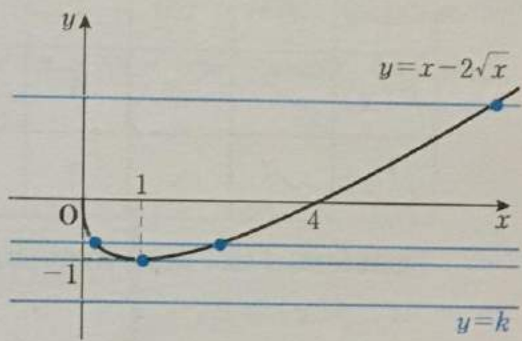
[Sol] Rearranging the equation, $x-2\sqrt{x}=k$

Let $f(x)=x-2\sqrt{x}$. The domain is $x \geqq 0$.

$$f'(x)=1-\frac{1}{\sqrt{x}}=\frac{\sqrt{x}-1}{\sqrt{x}}$$

When $f'(x)=0$ in $x>0$, $x=1$

x	0	$\cdots$	1	$\cdots$
$f'(x)$		$-$	0	$+$
$f(x)$	0	↘	-1	↗



Also, $\lim\limits_{x\to\infty} f(x)=\infty$

Therefore, the graph of $y=f(x)$ is as shown above.

From the common points of this graph and line $y=k$, the number of real solutions is:

When $-1<k\leqq 0$, 2 real solutions

When $k=-1$, $0<k$, 1 real solution

When $k<-1$, no real solutions

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O50b

2. Prove the following inequality and determine the value of x for which the equation holds true. ➡ O43

$$\frac{x^2}{2}+\frac{1}{x}\geqslant\frac{3}{2}\quad(x>0)$$

[Sol] Let $f(x)=\frac{x^2}{2}+\frac{1}{x}-\frac{3}{2}$.

$$f'(x)=x-\frac{1}{x^2}=\frac{(x-1)(x^2+x+1)}{x^2}$$

When $f'(x)=0$ in $x>0$, $x=1$

x	0	$\cdots$	1	$\cdots$
$f'(x)$	/	$-$	0	$+$
$f(x)$	/	↘	0	↗

Therefore, $f(x)$ has a minimum value of 0, at $x=1$.

$\therefore\ f(x)\geqslant 0$

Thus, when $x>0$, $\frac{x^2}{2}+\frac{1}{x}\geqslant\frac{3}{2}$

When $x=1$, the equation holds true.

3. Prove the following inequality. ➡ O44

$$x-1>\sqrt{x}\ln x\quad(x>1)$$

[Sol] Let $f(x)=x-1-\sqrt{x}\ln x$.

$$f'(x)=1-\left(\frac{1}{2\sqrt{x}}\cdot\ln x+\sqrt{x}\cdot\frac{1}{x}\right)=\frac{2\sqrt{x}-\ln x-2}{2\sqrt{x}}$$

$$f''(x)=\frac{\left(\frac{1}{\sqrt{x}}-\frac{1}{x}\right)\cdot 2\sqrt{x}-(2\sqrt{x}-\ln x-2)\cdot\frac{1}{\sqrt{x}}}{4x}=\frac{\ln x}{4x\sqrt{x}}$$

When $x>1$, $\ln x>0$; therefore, $f''(x)>0$

Thus, since $f'(x)$ increases in $x\geqslant 1$ and $f'(1)=0$, $f'(x)>0$

Therefore, since $f(x)$ increases in $x\geqslant 1$ and $f(1)=0$, $f(x)>0$

Thus, when $x>1$, $x-1>\sqrt{x}\ln x$

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The function which becomes $f(x)$ when differentiated is called the ***indefinite integral*** or ***antiderivative*** of $f(x)$ and is expressed as $\int f(x)dx$.
Let an indefinite integral of $f(x)$ be $F(x)$. Then, it is expressed as

$$\int f(x)dx=F(x)+C \quad (C \textit{ is the constant of integration})$$

Then, $f(x)$ is called the ***integrand*** and x is called the ***variable of integration***.
Also, the process of finding the indefinite integral of a function $f(x)$ is called ***integrating*** $f(x)$. Finding the indefinite integral is the reverse operation of differentiation. Since $(x^{\alpha+1})'=(\alpha+1)x^{\alpha}$ when α is a real number, and since $(\ln|x|)'=\frac{1}{x}$, the following formulas are true.

Indefinite Integral of x^{α}

$$\int x^{\alpha}dx=\frac{1}{\alpha+1}x^{\alpha+1}+C \qquad (\alpha \neq -1)$$

$$\int x^{-1}dx=\int \frac{1}{x}dx=\ln|x|+C$$

Find the following indefinite integrals, and then verify the answers by using differentiation.

Ex.

$$\int \frac{1}{x^4}dx=\int x^{-4}dx=-\frac{1}{3}x^{-3}+C=-\frac{1}{3x^3}+C$$

【Verification】 $\left(-\frac{1}{3x^3}+C\right)'=\left(-\frac{1}{3}x^{-3}+C\right)'=x^{-4}=\frac{1}{x^4}$

(1) $\int \frac{1}{x^5}dx=\int x^{-5}dx=-\frac{1}{4}x^{-4}+C=-\frac{1}{4x^4}+C$

【Verification】 $\left(-\frac{1}{4x^4}+C\right)'=\left(-\frac{1}{4}x^{-4}+C\right)'=x^{-5}=\frac{1}{x^5}$

【Reference】L I I I

In the case of **Ex.**, $-\frac{1}{3}x^{-3}+C$ is not a wrong answer. However, generally, the answer has to be written in the same form as the question. (For **Ex.**, use the fraction form such as $\frac{1}{x^4}$.)

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(2) $\int \frac{1}{x^2}dx = \int x^{-2}dx = -x^{-1}+C = -\frac{1}{x}+C$

【Verification】 $\left(-\frac{1}{x}+C\right)' = (-x^{-1}+C)' = x^{-2} = \frac{1}{x^2}$

(3) $\int \sqrt[3]{t^2}\,dt = \int t^{\frac{2}{3}}dt = \frac{3}{5}t^{\frac{5}{3}}+C = \frac{3}{5}t\sqrt[3]{t^2}+C$

【Verification】 $\left(\frac{3}{5}t\sqrt[3]{t^2}+C\right)' = \left(\frac{3}{5}t^{\frac{5}{3}}+C\right)' = t^{\frac{2}{3}} = \sqrt[3]{t^2}$

(4) $\int \frac{1}{u\sqrt{u}}du = \int u^{-\frac{3}{2}}du = -2u^{-\frac{1}{2}}+C = -\frac{2}{\sqrt{u}}+C$

【Verification】 $\left(-\frac{2}{\sqrt{u}}+C\right)' = \left(-2u^{-\frac{1}{2}}+C\right)' = u^{-\frac{3}{2}} = \frac{1}{u\sqrt{u}}$

(5) $\int (x+2)^4dx = \frac{1}{5}(x+2)^5+C$

【Verification】 $\left[\frac{1}{5}(x+2)^5+C\right]' = (x+2)^4$

(6) $\int \frac{1}{(x+2)^3}dx = \int (x+2)^{-3}dx = -\frac{1}{2}(x+2)^{-2}+C = -\frac{1}{2(x+2)^2}+C$

【Verification】 $\left[-\frac{1}{2(x+2)^2}+C\right]' = \left[-\frac{1}{2}(x+2)^{-2}+C\right]' = (x+2)^{-3} = \frac{1}{(x+2)^3}$

(7) $\int \frac{1}{x+2}dx = \ln|x+2|+C$

【Verification】 $(\ln|x+2|+C)' = \frac{1}{x+2}$

Indefinite Integrals 1

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Regarding indefinite integrals, the following identities are true.

Properties of Indefinite Integrals

$$\int kf(x)dx = k\int f(x)dx \qquad (k \text{ is a constant})$$

$$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$$

Find the following indefinite integrals.

Ex. $\int\left(\frac{2}{x^4}+1\right)dx = -\frac{2}{3x^3}+x+C \leftarrow \int\left(\frac{2}{x^4}+1\right)dx = 2\int x^{-4}dx + \int dx$

(1) $\int\left(3x^2+x-\frac{1}{x^3}\right)dx = x^3+\frac{1}{2}x^2+\frac{1}{2x^2}+C$

(2) $\int\left(4x^{\frac{1}{3}}+3x^{\frac{1}{2}}\right)dx = 3x^{\frac{4}{3}}+2x^{\frac{3}{2}}+C$

(3) $\int\left(y-\frac{1}{y}+1\right)dy = \frac{1}{2}y^2-\ln|y|+y+C$

$\int 1dx$ is also written as $\int dx$.

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(4) $\int\left(x+\frac{1}{x}\right)^2dx=\int\left(x^2+2+\frac{1}{x^2}\right)dx$

$=\frac{1}{3}x^3+2x-\frac{1}{x}+C$

(5) $\int\left(\sqrt{u}+\frac{1}{\sqrt{u}}\right)du=\int\left(u^{\frac{1}{2}}+u^{-\frac{1}{2}}\right)du$

$=\frac{2}{3}u^{\frac{3}{2}}+2u^{\frac{1}{2}}+C$

$=\frac{2}{3}u\sqrt{u}+2\sqrt{u}+C$

(6) $\int\sqrt{x}(x+1)dx=\int\left(x^{\frac{3}{2}}+x^{\frac{1}{2}}\right)dx$

$=\frac{2}{5}x^{\frac{5}{2}}+\frac{2}{3}x^{\frac{3}{2}}+C$

$=\frac{2}{5}x^2\sqrt{x}+\frac{2}{3}x\sqrt{x}+C$

(7) $\int(\sqrt[4]{x^3}+\sqrt[3]{x^4})dx=\int\left(x^{\frac{3}{4}}+x^{\frac{4}{3}}\right)dx$

$=\frac{4}{7}x^{\frac{7}{4}}+\frac{3}{7}x^{\frac{7}{3}}+C$

$=\frac{4}{7}x\sqrt[4]{x^3}+\frac{3}{7}x^2\sqrt[3]{x}+C$

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Let a and b be constants and an indefinite integral of a function $f(x)$ be $F(x)$.

$$[F(ax+b)]'=F'(ax+b)\cdot(ax+b)'=af(ax+b)$$

Integrating this equality with respect to x, then the following formula is true.

Indefinite Integral of $f(ax+b)$

When $F'(x)=f(x)$, $a\neq 0$,

$$\int f(ax+b)dx=\frac{1}{a}F(ax+b)+C$$

Find the following indefinite integrals.

Ex. $\int(2x-3)^4dx=\frac{1}{10}(2x-3)^5+C \leftarrow \int(2x-3)^4dx=\frac{1}{2}\cdot\frac{1}{5}(2x-3)^5+C$

(1) $\int(3x+1)^5dx=\frac{1}{18}(3x+1)^6+C$

(2) $\int(-2x+5)^{\frac{2}{3}}dx=-\frac{3}{10}(-2x+5)^{\frac{5}{3}}+C$

(3) $\int(4x-3)^{-\frac{5}{3}}dx=-\frac{3}{8}(4x-3)^{-\frac{2}{3}}+C$

(4) $\int\frac{dx}{2x+1}=\frac{1}{2}\ln|2x+1|+C$

$\int\frac{1}{f(x)}dx$ is also written as $\int\frac{dx}{f(x)}$.

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(5) $\int \sqrt{4x+1}\,dx = \int (4x+1)^{\frac{1}{2}}dx$

$$= \frac{1}{6}(4x+1)^{\frac{3}{2}} + C$$

$$= \frac{1}{6}(4x+1)\sqrt{4x+1} + C$$

(6) $\int \sqrt{4-x}\,dx = \int (4-x)^{\frac{1}{2}}dx$

$$= -\frac{2}{3}(4-x)^{\frac{3}{2}} + C$$

$$= -\frac{2}{3}(4-x)\sqrt{4-x} + C$$

(7) $\int \frac{dx}{\sqrt{2x+1}} = \int (2x+1)^{-\frac{1}{2}}dx$

$$= (2x+1)^{\frac{1}{2}} + C$$

$$= \sqrt{2x+1} + C$$

(8) $\int \frac{4}{4x-1}dx = \ln|4x-1| + C$

(9) $\int \frac{dx}{1-2x} = -\frac{1}{2}\ln|1-2x| + C$

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Find the following indefinite integrals.

Ex.

$$\int \frac{x^2-2x-3}{x^2}dx = \int\left(1-\frac{2}{x}-\frac{3}{x^2}\right)dx$$
$$= x - 2\ln|x| + \frac{3}{x} + C$$

(1) $$\int \frac{3x^2-2x+1}{x}dx = \int\left(3x-2+\frac{1}{x}\right)dx$$
$$= \frac{3}{2}x^2 - 2x + \ln|x| + C$$

(2) $$\int \frac{(x^2+1)^2}{x^2}dx = \int \frac{x^4+2x^2+1}{x^2}dx$$
$$= \int\left(x^2+2+\frac{1}{x^2}\right)dx$$
$$= \frac{1}{3}x^3 + 2x - \frac{1}{x} + C$$

(3) $$\int \frac{(2x-1)(x^2+1)}{x^3}dx = \int \frac{2x^3-x^2+2x-1}{x^3}dx$$
$$= \int\left(2-\frac{1}{x}+\frac{2}{x^2}-\frac{1}{x^3}\right)dx$$
$$= 2x - \ln|x| - \frac{2}{x} + \frac{1}{2x^2} + C$$

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Ex.

$$\int \frac{\sqrt{x}+1}{x}dx = \int \left(x^{-\frac{1}{2}} + \frac{1}{x}\right)dx$$

$$= 2x^{\frac{1}{2}} + \ln|x| + C$$

$$= 2\sqrt{x} + \ln x + C$$

Since the denominator of the integrand is x, $x \neq 0$
Since the integrand includes the term $\sqrt{x}$, $x \geqslant 0$
Therefore, the domain is $x > 0$.
Thus, the absolute value symbol can be removed as: $\ln|x| = \ln x$

(4) $$\int \frac{\sqrt{x}+\sqrt[3]{x}-1}{x}dx = \int \left(x^{-\frac{1}{2}} + x^{-\frac{2}{3}} - \frac{1}{x}\right)dx$$

$$= 2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} - \ln|x| + C$$

$$= 2\sqrt{x} + 3\sqrt[3]{x} - \ln x + C$$

(5) $$\int \frac{(1+\sqrt{x})^2}{x}dx = \int \frac{1+2\sqrt{x}+x}{x}dx$$

$$= \int \left(\frac{1}{x} + 2x^{-\frac{1}{2}} + 1\right)dx$$

$$= \ln|x| + 4x^{\frac{1}{2}} + x + C$$

$$= \ln x + 4\sqrt{x} + x + C$$

(6) $$\int \frac{(x+1)(x+2)}{x^2}dx = \int \frac{x^2+3x+2}{x^2}dx$$

$$= \int \left(1 + \frac{3}{x} + \frac{2}{x^2}\right)dx$$

$$= x + 3\ln|x| - \frac{2}{x} + C$$

Since the denominator of the integrand is x^2, $x \neq 0$
However, since the value is not necessarily $x > 0$, the absolute value symbol of $|x|$ cannot be removed.

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Find the following indefinite integrals.

(1) $$\int \frac{x^2+5x+12}{\sqrt{x}}dx=\int\left(x^{\frac{3}{2}}+5x^{\frac{1}{2}}+12x^{-\frac{1}{2}}\right)dx$$

$$=\frac{2}{5}x^{\frac{5}{2}}+\frac{10}{3}x^{\frac{3}{2}}+24x^{\frac{1}{2}}+C$$

$$=\frac{2}{5}x^2\sqrt{x}+\frac{10}{3}x\sqrt{x}+24\sqrt{x}+C$$

(2) $$\int\frac{x+\sqrt{x}}{\sqrt[3]{x}}dx=\int\left(x^{\frac{2}{3}}+x^{\frac{1}{6}}\right)dx$$

$$=\frac{3}{5}x^{\frac{5}{3}}+\frac{6}{7}x^{\frac{7}{6}}+C$$

$$=\frac{3}{5}x\sqrt[3]{x^2}+\frac{6}{7}x\sqrt[6]{x}+C$$

(3) $$\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2dx=\int\left(x+2+\frac{1}{x}\right)dx$$

$$=\frac{1}{2}x^2+2x+\ln|x|+C$$

$$=\frac{1}{2}x^2+2x+\ln x+C$$

Since the integrand includes the term $\frac{1}{\sqrt{x}}$, the domain is $x>0$. $\therefore\ |x|=x$

(4) $$\int\frac{(\sqrt{x}+1)^3}{\sqrt{x}}dx=\int\frac{x\sqrt{x}+3x+3\sqrt{x}+1}{\sqrt{x}}dx$$

$$=\int\left(x+3x^{\frac{1}{2}}+3+x^{-\frac{1}{2}}\right)dx$$

$$=\frac{1}{2}x^2+2x^{\frac{3}{2}}+3x+2x^{\frac{1}{2}}+C$$

$$=\frac{1}{2}x^2+2x\sqrt{x}+3x+2\sqrt{x}+C$$

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O55b

Ex.

$$\int \frac{dx}{\sqrt{x+2}+\sqrt{x}} = \int \frac{\sqrt{x+2}-\sqrt{x}}{(\sqrt{x+2}+\sqrt{x})(\sqrt{x+2}-\sqrt{x})}dx$$

$$= \frac{1}{2}\int \left[(x+2)^{\frac{1}{2}} - x^{\frac{1}{2}}\right]dx$$

$$= \frac{1}{2}\left[\frac{2}{3}(x+2)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right] + C$$

$$= \frac{1}{3}(x+2)\sqrt{x+2} - \frac{1}{3}x\sqrt{x} + C$$

(5) $$\int \frac{dx}{\sqrt{x}-\sqrt{x-3}} = \int \frac{\sqrt{x}+\sqrt{x-3}}{(\sqrt{x}-\sqrt{x-3})(\sqrt{x}+\sqrt{x-3})}dx$$

$$= \frac{1}{3}\int \left[x^{\frac{1}{2}} + (x-3)^{\frac{1}{2}}\right]dx$$

$$= \frac{1}{3}\left[\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{3}(x-3)^{\frac{3}{2}}\right] + C$$

$$= \mathbf{\frac{2}{9}x\sqrt{x} + \frac{2}{9}(x-3)\sqrt{x-3} + C}$$

(6) $$\int \frac{dx}{\sqrt{2-x}-\sqrt{1-x}} = \int \frac{\sqrt{2-x}+\sqrt{1-x}}{(\sqrt{2-x}-\sqrt{1-x})(\sqrt{2-x}+\sqrt{1-x})}dx$$

$$= \int \left[(2-x)^{\frac{1}{2}} + (1-x)^{\frac{1}{2}}\right]dx$$

$$= -\frac{2}{3}(2-x)^{\frac{3}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + C$$

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b)+C$$

$$= \mathbf{-\frac{2}{3}(2-x)\sqrt{2-x} - \frac{2}{3}(1-x)\sqrt{1-x} + C}$$

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Indefinite Integrals 1

Name

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Time : to :

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Find the following indefinite integrals.

Ex. $\int \frac{2x+1}{x-1}dx = \int \left(2+\frac{3}{x-1}\right)dx$

$= 2x + 3\ln|x-1| + C$

$$\begin{array}{r} 2 \\ x-1\overline{)2x+1} \\ \underline{2x-2} \\ 3 \end{array}$$

(1) $\int \frac{3x+2}{x+2}dx = \int \left(3-\frac{4}{x+2}\right)dx$

$= 3x - 4\ln|x+2| + C$

(2) $\int \frac{4x+1}{2x+1}dx = \int \left(2-\frac{1}{2x+1}\right)dx$

$= 2x - \frac{1}{2}\ln|2x+1| + C$

(3) $\int \frac{x^2+3}{x+1}dx = \int \left(x-1+\frac{4}{x+1}\right)dx$

$= \frac{1}{2}x^2 - x + 4\ln|x+1| + C$

$$\begin{array}{r} x\ -1 \\ x+1\overline{)x^2\quad +3} \\ \underline{x^2+x\quad} \\ -x+3 \\ \underline{-x-1} \\ 4 \end{array}$$

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In the case of **Ex.**, the expression can also be rearranged as $\frac{2x+1}{x-1} = \frac{2(x-1)+3}{x-1} = 2+\frac{3}{x-1}$.

056b

(4) $\int \frac{3x^2+4x-2}{3x+1}dx=\int\left(x+1-\frac{3}{3x+1}\right)dx$

$=\frac{1}{2}x^2+x-\ln|3x+1|+C$

(5) $\int \frac{x^3+2}{x-1}dx=\int\left(x^2+x+1+\frac{3}{x-1}\right)dx$

$=\frac{1}{3}x^3+\frac{1}{2}x^2+x+3\ln|x-1|+C$

(6) $\int \frac{x}{\sqrt{x+2}}dx=\int\left(\sqrt{x+2}-\frac{\boxed{2}}{\sqrt{x+2}}\right)dx$ ⟵ $\frac{x}{\sqrt{x+2}}=\frac{(x+2)-2}{\sqrt{x+2}}$ $=\sqrt{x+2}-\frac{2}{\sqrt{x+2}}$

$=\int\left[(x+2)^{\frac{1}{2}}-2(x+2)^{-\frac{1}{2}}\right]dx$

$=\frac{2}{3}(x+2)^{\frac{3}{2}}-4(x+2)^{\frac{1}{2}}+C$

$=\frac{2}{3}(x+2)\sqrt{x+2}-4\sqrt{x+2}+C$ $\left[=\frac{2}{3}(x-4)\sqrt{x+2}+C\right]$

(7) $\int \frac{2x}{\sqrt{2x+1}}dx=\int\left(\sqrt{2x+1}-\frac{1}{\sqrt{2x+1}}\right)dx$

$=\int\left[(2x+1)^{\frac{1}{2}}-(2x+1)^{-\frac{1}{2}}\right]dx$

$=\frac{1}{3}(2x+1)^{\frac{3}{2}}-(2x+1)^{\frac{1}{2}}+C$

$=\frac{1}{3}(2x+1)\sqrt{2x+1}-\sqrt{2x+1}+C$ $\left[=\frac{2}{3}(x-1)\sqrt{2x+1}+C\right]$

Indefinite Integrals 1

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Find the following indefinite integrals.

Ex.

$$\int \frac{x+5}{(x-3)(x+1)}dx$$

[Sol] Let $\frac{x+5}{(x-3)(x+1)}=\frac{a}{x-3}+\frac{b}{x+1}$.

Rearranging into an integrable form

$x+5=a(x+1)+b(x-3)$

Multiplying both sides by $(x-3)(x+1)$

$=(a+b)x+(a-3b)$

$\therefore \begin{cases} a+b=1 \\ a-3b=5 \end{cases}$

$\therefore\ a=2,\ b=-1$

※ $n\ln M=\ln M^n$

$\ln M-\ln N=\ln\frac{M}{N}$

Since $|x-3|^2=(x-3)^2$, the absolute value symbol can be removed.

Therefore,

$$\int \frac{x+5}{(x-3)(x+1)}dx=\int\left(\frac{2}{x-3}-\frac{1}{x+1}\right)dx$$

$$=2\ln|x-3|-\ln|x+1|+C$$

$$=\ln\frac{(x-3)^2}{|x+1|}+C$$

※

(1) $\int \frac{x-3}{(x-1)(x-2)}dx$

[Sol] Let $\frac{x-3}{(x-1)(x-2)}=\frac{a}{x-1}+\frac{b}{x-2}$.

$x-3=a(x-2)+b(x-1)$

$=(a+b)x-(2a+b)$

$\therefore \begin{cases} a+b=1 \\ 2a+b=3 \end{cases}$

$\therefore\ a=2,\ b=-1$

Therefore,

$$\int \frac{x-3}{(x-1)(x-2)}dx=\int\left(\frac{2}{x-1}-\frac{1}{x-2}\right)dx$$

$$=2\ln|x-1|-\ln|x-2|+C$$

$$=\ln\frac{(x-1)^2}{|x-2|}+C$$

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(2) $\int \frac{2x+7}{(x-1)(x+2)}dx$

[Sol] Let $\frac{2x+7}{(x-1)(x+2)}=\frac{a}{x-1}+\frac{b}{x+2}$.

$$2x+7=a(x+2)+b(x-1)$$
$$=(a+b)x+(2a-b)$$
$$\therefore \begin{cases} a+b=2 \\ 2a-b=7 \end{cases}$$
$$\therefore a=3,\ b=-1$$

Therefore,

$$\int \frac{2x+7}{(x-1)(x+2)}dx=\int\left(\frac{3}{x-1}-\frac{1}{x+2}\right)dx$$
$$=3\ln|x-1|-\ln|x+2|+C$$
$$=\ln\left|\frac{(x-1)^3}{x+2}\right|+C$$

Since $|x-1|^3 \neq (x-1)^3$, the absolute value symbol cannot be removed.

(3) $\int \frac{x+3}{2x^2-3x-2}dx$

[Sol] Let $\frac{x+3}{2x^2-3x-2}=\frac{x+3}{(x-2)(2x+1)}=\frac{a}{x-2}+\frac{b}{2x+1}$.

$$x+3=a(2x+1)+b(x-2)$$
$$=(2a+b)x+(a-2b)$$
$$\therefore \begin{cases} 2a+b=1 \\ a-2b=3 \end{cases}$$
$$\therefore a=1,\ b=-1$$

Therefore,

$$\int \frac{x+3}{2x^2-3x-2}dx=\int\left(\frac{1}{x-2}-\frac{1}{2x+1}\right)dx$$
$$=\ln|x-2|-\frac{1}{2}\ln|2x+1|+C$$
$$=\frac{1}{2}\ln\frac{(x-2)^2}{|2x+1|}+C \quad \left[=\ln\frac{|x-2|}{\sqrt{|2x+1|}}+C\right]$$

Indefinite Integrals 1

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Find the following indefinite integrals.

(1) $\displaystyle\int \frac{x^4}{x^2-1}\,dx$

[Sol] $\dfrac{x^4}{x^2-1}=x^2+1+\dfrac{1}{x^2-1}$

$$\begin{array}{r} x^2+1 \\ x^2-1\,\overline{)\,x^4} \\ \underline{x^4-x^2} \\ x^2 \\ \underline{x^2-1} \\ 1 \end{array}$$

Let $\dfrac{1}{x^2-1}=\dfrac{1}{(x-1)(x+1)}=\dfrac{a}{x-1}+\dfrac{b}{x+1}$.

$1=a(x+1)+b(x-1)$

$=(a+b)x+(a-b)$

$\therefore \begin{cases} a+b=0 \\ a-b=1 \end{cases}$

$\therefore\ a=\dfrac{1}{2},\ b=-\dfrac{1}{2}$

Therefore,

$$\int\frac{x^4}{x^2-1}dx=\int\left[x^2+1+\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)\right]dx$$

$$=\frac{1}{3}x^3+x+\frac{1}{2}(\ln|x-1|-\ln|x+1|)+C$$

$$=\frac{1}{3}x^3+x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$$

$$\left[=\frac{1}{3}x^3+x-\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|+C\right]$$

O58b

(2) $\int \frac{dx}{x(x+1)(x+2)}$

[Sol] Let $\frac{1}{x(x+1)(x+2)} = \frac{a}{x} + \frac{b}{x+1} + \frac{c}{x+2}$.

$$1 = a(x+1)(x+2) + bx(x+2) + cx(x+1)$$

$$= (a+b+c)x^2 + (3a+2b+c)x + 2a$$

$$\therefore \begin{cases} a+b+c=0 \\ 3a+2b+c=0 \\ 2a=1 \end{cases}$$

$$\therefore a = \frac{1}{2},\ b=-1,\ c=\frac{1}{2}$$

Therefore,

$$\int \frac{dx}{x(x+1)(x+2)} = \int \left[\frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)} \right] dx$$

$$= \frac{1}{2}\ln|x| - \ln|x+1| + \frac{1}{2}\ln|x+2| + C$$

$$= \frac{1}{2}\ln\frac{|x(x+2)|}{(x+1)^2} + C$$

$\ln M + \ln N = \ln MN$

$$\left[= \ln\frac{\sqrt{|x(x+2)|}}{|x+1|} + C \right]$$

O59a

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Find the following indefinite integrals.

(1) $\int \frac{2x+1}{(x+2)^2}\,dx$

[Sol] Let $\frac{2x+1}{(x+2)^2}=\frac{a}{x+2}+\frac{b}{(x+2)^2}$.

$$2x+1=a(x+2)+b$$
$$=ax+(2a+b)$$

$$\therefore \begin{cases} a=2 \\ 2a+b=1 \end{cases}$$

$$\therefore\ a=2,\ b=-3$$

Therefore,

$$\int \frac{2x+1}{(x+2)^2}\,dx=\int\left[\frac{2}{x+2}-\frac{3}{(x+2)^2}\right]dx$$
$$=2\ln|x+2|+\frac{3}{x+2}+C$$

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O59b

(2) $\displaystyle\int \frac{dx}{(x+1)(x+2)^2}$

[Sol] Let $\dfrac{1}{(x+1)(x+2)^2} = \dfrac{a}{x+1} + \dfrac{b}{x+2} + \dfrac{c}{(x+2)^2}$. ← ※

$$1 = a(x+2)^2 + b(x+1)(x+2) + c(x+1)$$
$$= (a+b)x^2 + (4a+3b+c)x + (4a+2b+c)$$
$$\therefore \begin{cases} a+b=0 \\ 4a+3b+c=0 \\ 4a+2b+c=1 \end{cases}$$
$$\therefore\ a=1,\ b=-1,\ c=-1$$

Therefore,

$$\int \frac{dx}{(x+1)(x+2)^2} = \int \left[\frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{(x+2)^2} \right] dx$$
$$= \ln|x+1| - \ln|x+2| + \frac{1}{x+2} + C$$
$$= \ln\left| \frac{x+1}{x+2} \right| + \frac{1}{x+2} + C$$

※ Let $\dfrac{1}{(x+1)(x+2)^2} = \dfrac{a}{x+1} + \dfrac{b}{(x+2)^2}$ ⋯①.

$$1 = a(x+2)^2 + b(x+1)$$
$$= ax^2 + (4a+b)x + (4a+b)$$
$$\therefore \begin{cases} a=0 \\ 4a+b=0 \\ 4a+b=1 \end{cases}$$

Since there are no values of a and b that satisfy the above conditions, the expression cannot be rearranged as ①.

O60a

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Indefinite Integrals 1

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Find the following indefinite integrals.

(1) $\int \sqrt{x}\left(3x^3-\frac{1}{x}\right)dx=\int\left(3x^{\frac{7}{2}}-x^{-\frac{1}{2}}\right)dx$

$=\frac{2}{3}x^{\frac{9}{2}}-2x^{\frac{1}{2}}+C$

$=\frac{2}{3}x^4\sqrt{x}-2\sqrt{x}+C$

➡ O52

(2) $\int\sqrt{1-5x}\,dx=\int(1-5x)^{\frac{1}{2}}dx$

$=-\frac{2}{15}(1-5x)^{\frac{3}{2}}+C$

$=-\frac{2}{15}(1-5x)\sqrt{1-5x}+C$

➡ O53

(3) $\int\frac{(\sqrt{x}+2)^3}{x}dx=\int\frac{x\sqrt{x}+6x+12\sqrt{x}+8}{x}dx$

$=\int\left(x^{\frac{1}{2}}+6+12x^{-\frac{1}{2}}+\frac{8}{x}\right)dx$

$=\frac{2}{3}x^{\frac{3}{2}}+6x+24x^{\frac{1}{2}}+8\ln|x|+C$

$=\frac{2}{3}x\sqrt{x}+6x+24\sqrt{x}+8\ln x+C$

➡ O54

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O60b

(4) $\displaystyle\int \frac{4x+1}{x+1}dx=\int\left(4-\frac{3}{x+1}\right)dx$ ➡ O56

$$=4x-3\ln|x+1|+C$$

(5) $\displaystyle\int \frac{x^2+x}{x+2}dx=\int\left(x-1+\frac{2}{x+2}\right)dx$ ➡ O56

$$=\frac{1}{2}x^2-x+2\ln|x+2|+C$$

(6) $\displaystyle\int \frac{3}{x^2-x-2}dx$ ➡ O57

[Sol] Let $\dfrac{3}{x^2-x-2}=\dfrac{3}{(x-2)(x+1)}=\dfrac{a}{x-2}+\dfrac{b}{x+1}$.

$$3=a(x+1)+b(x-2)$$
$$=(a+b)x+(a-2b)$$
$$\therefore \begin{cases} a+b=0 \\ a-2b=3 \end{cases}$$
$$\therefore\ a=1,\ b=-1$$

Therefore,

$$\int \frac{3}{x^2-x-2}dx=\int\left(\frac{1}{x-2}-\frac{1}{x+1}\right)dx$$
$$=\ln|x-2|-\ln|x+1|+C$$
$$=\ln\left|\frac{x-2}{x+1}\right|+C$$

Indefinite Integrals 2

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From the formulas of derivatives $(e^x)'=e^x$ and $(a^x)'=a^x\ln a$ $(a>0,\ a\neq 1)$, the following formulas are obtained.

Indefinite Integrals of Exponential Functions

$$\int e^x dx = e^x + C,\quad \int a^x dx = \frac{a^x}{\ln a} + C$$

Find the following indefinite integrals.

Ex. $\int e^{3x}dx=\frac{1}{3}e^{3x}+C$ ← $\int f(ax+b)dx=\frac{1}{a}F(ax+b)+C$

(1) $\int e^{2x}dx=\frac{1}{2}e^{2x}+C$

(2) $\int e^{-4x}dx=-\frac{1}{4}e^{-4x}+C$

(3) $\int 2e^{2x}dx=e^{2x}+C$

(4) $\int e^{-\frac{1}{2}x}dx=-2e^{-\frac{1}{2}x}+C$

Ex. $\int 2^x dx=\frac{2^x}{\ln 2}+C$

(5) $\int 3^x dx=\frac{3^x}{\ln 3}+C$

(6) $\int 5^{-x}dx=-\frac{5^{-x}}{\ln 5}+C$

(7) $\int 7^{2x-1}dx=\frac{7^{2x-1}}{2\ln 7}+C$ ← $\int f(ax+b)dx=\frac{1}{a}F(ax+b)+C$

(8) $\int \left(\frac{1}{3}\right)^x dx=\frac{\left(\frac{1}{3}\right)^x}{\ln\frac{1}{3}}+C$

$\left[=-\frac{\left(\frac{1}{3}\right)^x}{\ln 3}+C\right]$ $\left[=-\frac{1}{3^x\ln 3}+C\right]$

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O61b

(9) $\int \frac{1-e^{2x}}{1+e^x}dx = \int \frac{(1+e^x)(\boxed{1-e^x})}{1+e^x}dx$

$= \int (1-e^x)dx$

$= x - e^x + C$

(10) $\int \frac{e^{2x}-e^{-2x}}{e^x+e^{-x}}dx = \int \frac{(e^x+e^{-x})(e^x-e^{-x})}{e^x+e^{-x}}dx$

$= \int (e^x - e^{-x})dx$

$= e^x + e^{-x} + C$

(11) $\int \left(\frac{e^x+e^{-x}}{2}\right)^2 dx = \frac{1}{4}\int (e^{2x}+2+e^{-2x})dx$

$= \frac{1}{4}\left(\frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}\right)+C$

$= \frac{1}{8}e^{2x}+\frac{1}{2}x-\frac{1}{8}e^{-2x}+C$

(12) $\int \frac{3^{3x}-1}{3^x-1}dx = \int \frac{(3^x-1)(3^{2x}+3^x+1)}{3^x-1}dx$ ⟵ a^3-b^3 $=(a-b)(a^2+ab+b^2)$

$= \int (3^{2x}+3^x+1)dx$

$= \frac{3^{2x}}{2\ln 3}+\frac{3^x}{\ln 3}+x+C$

Indefinite Integrals 2

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From the formulas of derivatives $(\sin x)'=\cos x$, $(\cos x)'=-\sin x$, $(\tan x)'=\dfrac{1}{\cos^2 x}$ and $\left(\dfrac{1}{\tan x}\right)'=-\dfrac{1}{\sin^2 x}$, the following formulas are obtained.

Indefinite Integrals of Trigonometric Functions

$$\int \sin x\,dx=-\cos x+C,\quad \int \cos x\,dx=\sin x+C$$

$$\int \frac{1}{\cos^2 x}dx=\tan x+C,\quad \int \frac{1}{\sin^2 x}dx=-\frac{1}{\tan x}+C$$

Find the following indefinite integrals.

Ex. $\int(\sin x-2\cos x)dx=-\cos x-2\sin x+C$

(1) $\int(\sin x+\cos x)dx=-\cos x+\sin x+C$

(2) $\int(3\cos x-\sin x)dx=3\sin x+\cos x+C$

(3) $\int\frac{3+\sin^3 x}{\sin^2 x}dx=\int\left(\frac{3}{\sin^2 x}+\sin x\right)dx$

$=-\frac{3}{\tan x}-\cos x+C$

(4) $\int\frac{1-\cos^3 x}{\cos^2 x}dx=\int\left(\frac{1}{\cos^2 x}-\cos x\right)dx$

$=\tan x-\sin x+C$

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O62b

(5) $\int \frac{x-\sin^2 x}{x\sin^2 x}dx=\int\left(\frac{1}{\sin^2 x}-\frac{1}{x}\right)dx$

$=-\frac{1}{\tan x}-\ln|x|+C$

(6) $\int \frac{\sin^2 x}{1-\cos x}dx=\int\frac{1-\cos^2 x}{1-\cos x}dx$

$=\int\frac{(1+\cos x)(1-\cos x)}{1-\cos x}dx$

$=\int(1+\cos x)dx$

$=x+\sin x+C$

(7) $\int \tan^2 x\,dx=\int\frac{\sin^2 x}{\cos^2 x}dx$

$=\int\frac{1-\cos^2 x}{\cos^2 x}dx$

$=\int\left(\frac{1}{\cos^2 x}-1\right)dx$

$=\tan x-x+C$

(8) $\int\frac{dx}{\tan^2 x}=\int\frac{\cos^2 x}{\sin^2 x}dx$

$=\int\frac{1-\sin^2 x}{\sin^2 x}dx$

$=\int\left(\frac{1}{\sin^2 x}-1\right)dx$

$=-\frac{1}{\tan x}-x+C$

Indefinite Integrals 2

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Find the following indefinite integrals.

Ex. $\int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$ ⟵ $\int f(ax+b)dx = \frac{1}{a}F(ax+b)+C$

(1) $\int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$

(2) $\int \cos(-2x+1)dx = -\frac{1}{2}\sin(-2x+1)+C$

(3) $\int \sin(1-x)dx = \cos(1-x)+C$

(4) $\int \frac{dx}{\cos^2 2x} = \frac{1}{2}\tan 2x + C$

(5) $\int \frac{dx}{\sin^2 3x} = -\frac{1}{3\tan 3x}+C$

O63b

(6) $\int(\sin 3\theta-\cos 5\theta)\,d\theta=-\frac{1}{3}\cos 3\theta-\frac{1}{5}\sin 5\theta+C$

(7) $\int \sin\frac{x+1}{2}\,dx=-2\cos\frac{x+1}{2}+C$

(8) $\int \cos\left(3-\frac{x}{2}\right)dx=-2\sin\left(3-\frac{x}{2}\right)+C$

(9) $\int \sin\pi t\,dt=-\frac{1}{\pi}\cos\pi t+C$

(10) $\int \cos\frac{\pi}{2}t\,dt=\frac{2}{\pi}\sin\frac{\pi}{2}t+C$

(11) $\int \sin\left(2x-\frac{\pi}{4}\right)dx=-\frac{1}{2}\cos\left(2x-\frac{\pi}{4}\right)+C$

Indefinite Integrals 2

Name ______

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Find the following indefinite integrals.

Ex.

$$\int \sin^2 x\,dx = \frac{1}{2}\int(1-\cos 2x)\,dx \quad \Leftarrow \quad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

(1) $$\int \cos^2 x\,dx = \frac{1}{2}\int(1+\cos 2x)\,dx \quad \Leftarrow \quad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

(2) $$\int \sin x \cos x\,dx = \frac{1}{2}\int \sin 2x\,dx \quad \Leftarrow \quad 2\sin\alpha\cos\alpha = \sin 2\alpha$$

$$= -\frac{1}{4}\cos 2x + C$$

(3) $$\int \cos^2\frac{x}{2}\,dx = \frac{1}{2}\int(1+\cos x)\,dx$$

$$= \frac{1}{2}(x+\sin x) + C \quad \left[= \frac{1}{2}x + \frac{1}{2}\sin x + C\right]$$

$\cos 2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

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(4) $\int(\sin x+\cos x)^2dx=\int(\sin^2x+2\sin x\cos x+\cos^2x)dx$

$$=\int(1+\sin 2x)dx$$

$$=x-\frac{1}{2}\cos 2x+C$$

(5) $\int \sin x(1+\cos x)dx=\int(\sin x+\sin x\cos x)dx$

$$=\int\left(\sin x+\frac{1}{2}\sin 2x\right)dx$$

$$=-\cos x-\frac{1}{4}\cos 2x+C$$

(6) $\int\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2dx=\int\left(\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}+\cos^2\frac{x}{2}\right)dx$

$$=\int(1+\sin x)dx$$

$$=x-\cos x+C$$

(7) $\int\frac{dx}{1+\cos 2x}=\frac{1}{2}\int\frac{dx}{\cos^2x}$

$$=\frac{1}{2}\tan x+C$$

O65a

KUMON

Indefinite Integrals 2

Name

Date / /

Time : to :

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(mistakes) 0	1	2	3	4~

Find the following indefinite integrals.

(1) $$\int\left(\sin x+\frac{1}{\sin x}\right)^2dx=\int\left(\sin^2 x+2+\frac{1}{\sin^2 x}\right)dx$$
$$=\int\left[\frac{1}{2}(1-\cos 2x)+2+\frac{1}{\sin^2 x}\right]dx$$
$$=\int\left(\frac{5}{2}-\frac{1}{2}\cos 2x+\frac{1}{\sin^2 x}\right)dx$$
$$=\frac{5}{2}x-\frac{1}{4}\sin 2x-\frac{1}{\tan x}+C$$

(2) $$\int\left(\cos x+\frac{1}{\cos x}\right)^2dx=\int\left(\cos^2 x+2+\frac{1}{\cos^2 x}\right)dx$$
$$=\int\left[\frac{1}{2}(1+\cos 2x)+2+\frac{1}{\cos^2 x}\right]dx$$
$$=\int\left(\frac{5}{2}+\frac{1}{2}\cos 2x+\frac{1}{\cos^2 x}\right)dx$$
$$=\frac{5}{2}x+\frac{1}{4}\sin 2x+\tan x+C$$

(3) $$\int\left(\tan x+\frac{1}{\tan x}\right)^2dx=\int\left(\tan^2 x+2+\frac{1}{\tan^2 x}\right)dx$$
$$=\int\left(\frac{\sin^2 x}{\cos^2 x}+2+\frac{\cos^2 x}{\sin^2 x}\right)dx$$
$$=\int\left(\frac{1-\cos^2 x}{\cos^2 x}+2+\frac{1-\sin^2 x}{\sin^2 x}\right)dx$$
$$=\int\left(\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x}\right)dx$$
$$=\tan x-\frac{1}{\tan x}+C$$

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(4) $\int \sin^4 x\,dx = \int (\sin^2 x)^2 dx$

$$= \int \left[\frac{1}{2}(1-\cos 2x)\right]^2 dx$$

$$= \frac{1}{4}\int (1-2\cos 2x+\cos^2 2x)\,dx$$

$$= \frac{1}{4}\int \left[1-2\cos 2x+\frac{1}{2}(1+\cos 4x)\right]dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2}-2\cos 2x+\frac{1}{2}\cos 4x\right)dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x-\sin 2x+\frac{1}{8}\sin 4x\right)+C$$

$$= \frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$$

(5) $\int (\sqrt{3}\sin x+\cos x)^2 dx = \int (3\sin^2 x+2\sqrt{3}\sin x\cos x+\cos^2 x)\,dx$

$$= \int \left[3\cdot\frac{1}{2}(1-\cos 2x)+\sqrt{3}\sin 2x+\frac{1}{2}(1+\cos 2x)\right]dx$$

$$= \int (2-\cos 2x+\sqrt{3}\sin 2x)\,dx$$

$$= 2x-\frac{1}{2}\sin 2x-\frac{\sqrt{3}}{2}\cos 2x+C$$

Alternative Solution

$\int (\sqrt{3}\sin x+\cos x)^2 dx = \int (3\sin^2 x+2\sqrt{3}\sin x\cos x+\cos^2 x)\,dx$

$$= \int (1+2\sin^2 x+\sqrt{3}\sin 2x)\,dx$$

$\sin^2 x+\cos^2 x=1$

$$= \int \left[1+2\cdot\frac{1}{2}(1-\cos 2x)+\sqrt{3}\sin 2x\right]dx$$

$$= \int (2-\cos 2x+\sqrt{3}\sin 2x)\,dx$$

$$= 2x-\frac{1}{2}\sin 2x-\frac{\sqrt{3}}{2}\cos 2x+C$$

Indefinite Integrals 2

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Find the following indefinite integrals.

Ex.

$$\int \cos^3 x\,dx = \frac{1}{4}\int (3\cos x + \cos 3x)\,dx \quad \leftarrow \quad \cos^3\alpha = \frac{1}{4}(3\cos\alpha + \cos 3\alpha)$$

$$= \frac{1}{4}\left(3\sin x + \frac{1}{3}\sin 3x\right) + C$$

$$= \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + C$$

(1) $$\int \sin^3 x\,dx = \frac{1}{4}\int (3\sin x - \sin 3x)\,dx \quad \leftarrow \quad \sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha)$$

$$= \frac{1}{4}\left(-3\cos x + \frac{1}{3}\cos 3x\right) + C$$

$$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$$

(2) $$\int \cos^3 2x\,dx = \frac{1}{4}\int (3\cos 2x + \cos 6x)\,dx$$

$$= \frac{1}{4}\left(\frac{3}{2}\sin 2x + \frac{1}{6}\sin 6x\right) + C$$

$$= \frac{3}{8}\sin 2x + \frac{1}{24}\sin 6x + C$$

(3) $$\int \sin^3(2x-1)\,dx = \frac{1}{4}\int [3\sin(2x-1) - \sin(6x-3)]\,dx$$

$$= \frac{1}{4}\left[-\frac{3}{2}\cos(2x-1) + \frac{1}{6}\cos(6x-3)\right] + C$$

$$= -\frac{3}{8}\cos(2x-1) + \frac{1}{24}\cos(6x-3) + C$$

$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$
$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

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Ex.

$$\int \cos 4x \sin 3x\,dx = \frac{1}{2}\int (\sin 7x - \sin x)\,dx$$

$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

$$= \frac{1}{2}\left(-\frac{1}{7}\cos 7x + \cos x\right) + C$$

$$= -\frac{1}{14}\cos 7x + \frac{1}{2}\cos x + C$$

(4) $$\int \sin 3x \cos x\,dx = \frac{1}{2}\int (\sin 4x + \sin 2x)\,dx$$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$$= \frac{1}{2}\left(-\frac{1}{4}\cos 4x - \frac{1}{2}\cos 2x\right) + C$$

$$= -\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + C$$

(5) $$\int \sin 3x \sin 2x\,dx = -\frac{1}{2}\int (\cos 5x - \cos x)\,dx$$

$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$

$$= -\frac{1}{2}\left(\frac{1}{5}\sin 5x - \sin x\right) + C$$

$$= -\frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C$$

(6) $$\int \cos 2x \cos 3x\,dx = \frac{1}{2}\int [\cos 5x + \cos(-x)]\,dx$$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$

$$= \frac{1}{2}\int (\cos 5x + \cos x)\,dx$$

$\cos(-\theta) = \cos\theta$

$$= \frac{1}{2}\left(\frac{1}{5}\sin 5x + \sin x\right) + C$$

$$= \frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C$$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$

$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$

Indefinite Integrals 2

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Find the function $f(x)$ which satisfies the following conditions.

Ex. $f'(x)=x\sqrt{x},\ f(1)=-\frac{3}{5}$

[Sol] $f(x)=\int x\sqrt{x}\,dx=\int x^{\frac{3}{2}}dx=\frac{2}{5}x^{\frac{5}{2}}+C=\frac{2}{5}x^2\sqrt{x}+C$

Since $f(1)=-\frac{3}{5}$, $C=-1$ ⟵ $\boxed{f(1)=\frac{2}{5}\cdot1^2\cdot\sqrt{1}+C=\frac{2}{5}+C}$

$\therefore\ f(x)=\frac{2}{5}x^2\sqrt{x}-1$

(1) $f'(x)=e^x-1,\ f(1)=e$

[Sol] $f(x)=\int(e^x-1)dx=e^x-x+C$

Since $f(1)=e$, $C=1$

$\therefore\ f(x)=e^x-x+1$

(2) $f'(x)=\frac{1-x-x^2}{x^2},\ f(e)=-1$

[Sol] $f(x)=\int\frac{1-x-x^2}{x^2}dx$

$=\int\left(\frac{1}{x^2}-\frac{1}{x}-1\right)dx$

$=-\frac{1}{x}-\ln|x|-x+C$

Since $f(e)=-1$, $C=e+\frac{1}{e}$

$\therefore\ f(x)=-\frac{1}{x}-\ln|x|-x+e+\frac{1}{e}$

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(3) $f''(x)=-2x+3,\ f(0)=1,\ f'(0)=-2$

[Sol] $f'(x)=\int(-2x+3)dx=-x^2+3x+C_1$ ← Since $f(x)$ also needs to be found, use C_1 instead of C.

Since $f'(0)=-2$, $C_1=\boxed{-2}$

$\therefore\ f'(x)=-x^2+3x-2$

$\therefore\ f(x)=\int(-x^2+3x-2)dx=-\frac{1}{3}x^3+\frac{3}{2}x^2-2x+C_2$

Since $f(0)=1$, $C_2=1$

$\therefore\ f(x)=-\frac{1}{3}x^3+\frac{3}{2}x^2-2x+1$

(4) $f''(x)=\frac{1}{\sqrt{x}},\ f(0)=0,\ f'(1)=1$

[Sol] $f'(x)=\int\frac{1}{\sqrt{x}}dx=\int x^{-\frac{1}{2}}dx=2x^{\frac{1}{2}}+C_1=2\sqrt{x}+C_1$

Since $f'(1)=1$, $C_1=-1$

$\therefore\ f'(x)=2\sqrt{x}-1$

$\therefore\ f(x)=\int\left(2x^{\frac{1}{2}}-1\right)dx=\frac{4}{3}x^{\frac{3}{2}}-x+C_2=\frac{4}{3}x\sqrt{x}-x+C_2$

Since $f(0)=0$, $C_2=0$

$\therefore\ f(x)=\frac{4}{3}x\sqrt{x}-x$

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Indefinite Integrals 2

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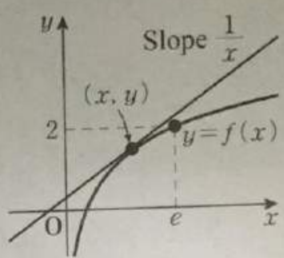
Ex. Let $f(x)$ be a differentiable function which is defined for $x>0$. Given that the slope of the tangent at point (x, y) on the curve $y=f(x)$ is $\frac{1}{x}$, find the curve which passes through point $(e, 2)$.

[Sol] Since the slope of the tangent at point (x, y) on the curve $y=f(x)$ is $f'(x)$,



$$f'(x)=\frac{1}{x}$$

$$\therefore\ f(x)=\int\frac{1}{x}dx$$

$$=\ln|x|+C$$

$$=\ln x+C$$

Since $x>0$, $|x|=x$

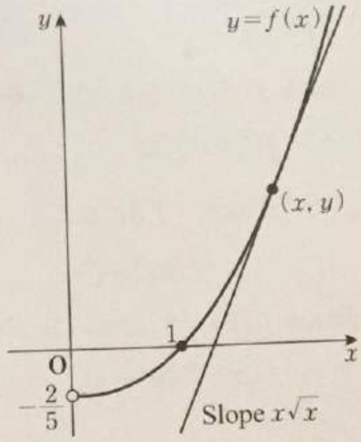
Since $f(e)=2$, $C=1$

$f(e)=\ln e+C=1+C$

$$\therefore\ f(x)=\ln x+1$$

1. Let $f(x)$ be a differentiable function which is defined for $x>0$. Given that the slope of the tangent at point (x, y) on the curve $y=f(x)$ is $x\sqrt{x}$, find the curve which passes through point $(1, 0)$.

[Sol] Since the slope of the tangent at point (x, y) on the curve $y=f(x)$ is $f'(x)$,



$$f'(x)=x\sqrt{x}$$

$$\therefore\ f(x)=\int x\sqrt{x}\,dx$$

$$=\int x^{\frac{3}{2}}dx$$

$$=\frac{2}{5}x^{\frac{5}{2}}+C$$

$$=\frac{2}{5}x^2\sqrt{x}+C$$

Since $f(1)=0$, $C=-\frac{2}{5}$

$$\therefore\ f(x)=\frac{2}{5}x^2\sqrt{x}-\frac{2}{5}$$

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2. Let $f(x)$ be a differentiable function. Given that the slope of the normal at point (x, y) on the curve $y=f(x)$ is 3^x, find the curve which passes through the origin.

[Sol] Since the slope of the normal at point (x, y) on the curve $y=f(x)$

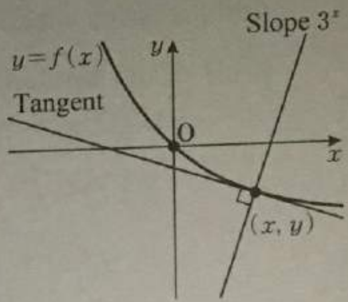
is $-\dfrac{1}{f'(x)}$, ← 03

$$-\frac{1}{f'(x)}=3^x$$

$$\therefore\ f'(x)=-3^{-x}$$

$$\therefore\ f(x)=\int(-3^{-x})dx$$

$$=\frac{3^{-x}}{\ln 3}+C$$



Since $f(0)=0$, $C=-\dfrac{1}{\ln 3}$

$$\therefore\ f(x)=\frac{3^{-x}}{\ln 3}-\frac{1}{\ln 3}\quad\left[=\frac{1}{3^x\ln 3}-\frac{1}{\ln 3}\right]\quad\left[=\frac{\left(\frac{1}{3}\right)^x}{\ln 3}-\frac{1}{\ln 3}\right]$$

3. Let $f(x)$ be a differentiable function. Given that the slope of the tangent at point (x, y) on the curve $y=f(x)$ is ke^x (k is a constant), find the curve which its equation of the tangent is $y=3x+2$ at point $(0, f(0))$.

[Sol] Since the slope of the tangent at point (x, y) on the curve $y=f(x)$ is $f'(x)$,

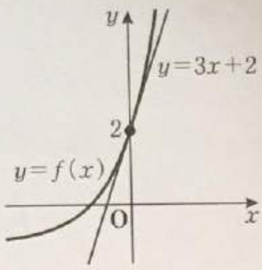
$$f'(x)=ke^x$$

Since $f'(0)=3$, $k=3$ ← Since the slope of the tangent at point $(0, f(0))$ is 3, $f'(0)=3$

$$\therefore\ f'(x)=3e^x$$

$$\therefore\ f(x)=\int 3e^x dx$$

$$=3e^x+C$$



Since $f(0)=2$, $C=-1$ ← Substituting $x=0$ into $y=3x+2$, $y=2$. Therefore, the curve $y=f(x)$ passes through point $(0, 2)$.

$$\therefore\ f(x)=3e^x-1$$

Indefinite Integrals 2

Name

Date / /

Time : to :

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1. Given that an indefinite integral $F(x)$ of the function $f(x)$ is equal to $xf(x)-\dfrac{1}{x}$ and when $f\left(\dfrac{1}{\sqrt{2}}\right)=1$, find $f(x)$. $(x>0)$

[Sol] Differentiating both sides of $F(x)=xf(x)-\dfrac{1}{x}$ with respect to x,

$$f(x)=1\cdot f(x)+xf'(x)+\frac{1}{x^2}$$

$$=f(x)+xf'(x)+\frac{1}{x^2}$$

Since an indefinite integral of $f(x)$ is $F(x)$, $F'(x)=f(x)$

$$\therefore\ f'(x)=-\frac{1}{x^3}$$

$$\therefore\ f(x)=\int\left(-\frac{1}{x^3}\right)dx=\frac{1}{2x^2}+C$$

Since $f\left(\dfrac{1}{\sqrt{2}}\right)=1$, $C=0$

$$\therefore\ f(x)=\frac{1}{2x^2}$$

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2. Find the function $f(x)$ which satisfies the following conditions.

$$f'(x)=\frac{1}{x^2+3x+2},\ f(0)=0$$

[Sol] $f(x)=\int\frac{dx}{x^2+3x+2}=\int\frac{dx}{(x+1)(x+2)}$ ← 057

Let $\frac{1}{(x+1)(x+2)}=\frac{a}{x+1}+\frac{b}{x+2}$.

Multiplying both sides by $(x+1)(x+2)$

$$\begin{aligned}1&=a(x+2)+b(x+1)\\&=(a+b)x+(2a+b)\end{aligned}$$

$$\therefore \begin{cases}a+b=0\\2a+b=1\end{cases}$$

$\therefore\ a=1,\ b=-1$

Therefore,

$$\begin{aligned}f(x)&=\int\left(\frac{1}{x+1}-\frac{1}{x+2}\right)dx\\&=\ln|x+1|-\ln|x+2|+C\\&=\ln\left|\frac{x+1}{x+2}\right|+C\end{aligned}$$

Since $f(0)=0$, $C=\ln 2$

Thus,

$$\begin{aligned}f(x)&=\ln\left|\frac{x+1}{x+2}\right|+\ln 2\\&=\ln\left|\frac{2(x+1)}{x+2}\right|\end{aligned}$$

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Indefinite Integrals 2

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1. Find the following indefinite integrals.

➡ O61

(1) $$\int \frac{e^{2x}-1}{e^x-1}dx=\int \frac{(e^x+1)(e^x-1)}{e^x-1}dx$$
$$=\int (e^x+1)dx$$
$$=e^x+x+C$$

➡ O62

(2) $$\int \frac{1+\cos^3 x}{\cos^2 x}dx=\int\left(\frac{1}{\cos^2 x}+\cos x\right)dx$$
$$=\tan x+\sin x+C$$

➡ O63

(3) $$\int(\sin 5x+\cos 3x)dx=-\frac{1}{5}\cos 5x+\frac{1}{3}\sin 3x+C$$

➡ O64

(4) $$\int(1-\cos x)^2dx=\int(1-2\cos x+\cos^2 x)dx$$
$$=\int\left[1-2\cos x+\frac{1}{2}(1+\cos 2x)\right]dx$$
$$=\int\left(\frac{3}{2}-2\cos x+\frac{1}{2}\cos 2x\right)dx$$
$$=\frac{3}{2}x-2\sin x+\frac{1}{4}\sin 2x+C$$

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(5) $\int \cos 6x \sin 3x\,dx = \frac{1}{2}\int(\sin 9x - \sin 3x)\,dx$ ➡ O66

$= \frac{1}{2}\left(-\frac{1}{9}\cos 9x + \frac{1}{3}\cos 3x\right) + C$

$= -\frac{1}{18}\cos 9x + \frac{1}{6}\cos 3x + C$

2. Find the function $f(x)$ which satisfies $f'(x) = \sin x + \cos x$ and $f\left(\frac{\pi}{2}\right) = 0$.

➡ O67

[Sol] $f(x) = \int(\sin x + \cos x)\,dx = -\cos x + \sin x + C$

Since $f\left(\frac{\pi}{2}\right) = 0$, $C = -1$

$\therefore\ f(x) = -\cos x + \sin x - 1$

3. Let $f(x)$ be a differentiable function. Given that the slope of the tangent at point (x, y) on the curve $y = f(x)$ is 2^x, find the curve which passes through the origin. ➡ O68

[Sol] Since the slope of the tangent at point (x, y) on the curve $y = f(x)$ is $f'(x)$,

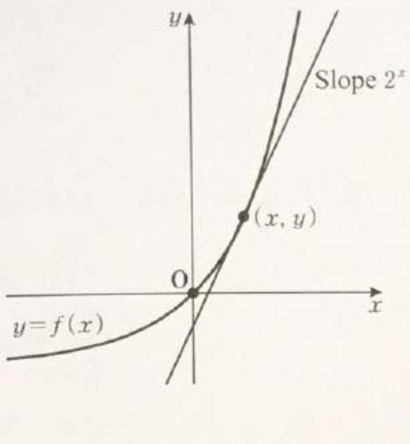
$f'(x) = 2^x$

$\therefore\ f(x) = \int 2^x\,dx$

$= \frac{2^x}{\ln 2} + C$

Since $f(0) = 0$, $C = -\frac{1}{\ln 2}$

$\therefore\ f(x) = \frac{2^x}{\ln 2} - \frac{1}{\ln 2}$



Integration by Substitution

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Let $F(x)=\int f(x)dx$ ···①. If a function of t, $x=g(t)$, is substituted into x, $F(x)=F(g(t))$ is also said to be a function of t. Given that $g(t)$ is a differentiable function, if $F(x)$ is differentiated with respect to t using Chain Rule I in N151, then

$$\frac{d}{dt}F(x)=\frac{d}{dx}F(x)\cdot\frac{dx}{dt}=f(x)g'(t)=f(g(t))g'(t)$$

$$\therefore\ F(x)=\int f(g(t))g'(t)dt \quad \cdots②$$

From ① and ②, the following formula is true.

Integration by Substitution I

$$\int f(x)dx=\int f(g(t))g'(t)dt \qquad (x=g(t))$$

This method is called ***Integration by Substitution***.

Find the following indefinite integrals using Integration by Substitution.

Ex. $\int(2x-5)^3dx$

[Sol] Let $2x-5=t$. Since $x=\frac{t+5}{2}$, $dx=\frac{1}{2}dt$ ← Since $\frac{dx}{dt}=\frac{1}{2}$

$$\therefore\ \int(2x-5)^3dx=\int t^3\cdot\frac{1}{2}dt$$ ← $dx=\frac{1}{2}dt$

$$=\frac{1}{2}\int t^3dt$$

$\int t^3dt=\frac{1}{4}t^4+C$

$$=\boxed{\frac{1}{8}t^4}+C$$

Substituting $2x-5$ into t

$$=\boxed{\frac{1}{8}(2x-5)^4}+C$$

Answers: $\frac{1}{8}t^4$, $\frac{1}{8}(2x-5)^4$

$\frac{dx}{dt}=\frac{1}{2}$ can also be expressed as $dx=\frac{1}{2}dt$.

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(1) $\int (3x+2)^5 dx$

[Sol] Let $3x+2=t$. Since $x=\frac{t-2}{3}$, $dx=\frac{1}{3}dt$

$$\begin{aligned}\therefore \int (3x+2)^5 dx &= \int t^5 \cdot \frac{1}{3}dt \\ &= \frac{1}{3}\int t^5 dt \\ &= \frac{1}{18}t^6+C \\ &= \frac{1}{18}(3x+2)^6+C\end{aligned}$$

(2) $\int \frac{dx}{3x-1}$

[Sol] Let $3x-1=t$. Since $x=\frac{t+1}{3}$, $dx=\frac{1}{3}dt$

$$\begin{aligned}\therefore \int \frac{dx}{3x-1} &= \int \frac{1}{t}\cdot\frac{1}{3}dt \\ &= \frac{1}{3}\int \frac{dt}{t} \\ &= \frac{1}{3}\ln|t|+C \\ &= \frac{1}{3}\ln|3x-1|+C\end{aligned}$$

(3) $\int \frac{x}{(x-1)^3}dx$

[Sol] Let $x-1=t$. Since $x=t+1$, $dx=dt$

$$\begin{aligned}\therefore \int \frac{x}{(x-1)^3}dx &= \int \frac{t+1}{t^3}dt \\ &= \int \left(\frac{1}{t^2}+\frac{1}{t^3}\right)dt \\ &= -\frac{1}{t}-\frac{1}{2t^2}+C \\ &= -\frac{2t+1}{2t^2}+C \\ &= -\frac{2x-1}{2(x-1)^2}+C \quad \left[=-\frac{1}{x-1}-\frac{1}{2(x-1)^2}+C\right]\end{aligned}$$

Integration by Substitution

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Time : to :

Find the following indefinite integrals using Integration by Substitution.

Ex. $\int(x-2)\sqrt{3-x}\,dx$

[Sol] Let $\sqrt{3-x}=t$. Since $3-x=t^2$, $x=3-t^2$

$\therefore\ dx=-2t\,dt$

$$\therefore \int(x-2)\sqrt{3-x}\,dx=\int(1-t^2)t\cdot(-2t)\,dt$$
$$=2\int(t^4-t^2)\,dt$$
$$=2\left(\frac{1}{5}t^5-\frac{1}{3}t^3\right)+C$$
$$=\frac{2}{15}t^3(3t^2-5)+C$$
$$=\frac{2}{15}(4-3x)(3-x)\sqrt{3-x}+C$$

(1) $\int(x+1)\sqrt{1-x}\,dx$

[Sol] Let $\sqrt{1-x}=t$. Since $1-x=t^2$, $x=1-t^2$

$\therefore\ dx=-2t\,dt$

$$\therefore \int(x+1)\sqrt{1-x}\,dx=\int(2-t^2)t\cdot(-2t)\,dt$$
$$=2\int(t^4-2t^2)\,dt$$
$$=2\left(\frac{1}{5}t^5-\frac{2}{3}t^3\right)+C$$
$$=\frac{2}{15}t^3(3t^2-10)+C$$
$$=-\frac{2}{15}(3x+7)(1-x)\sqrt{1-x}+C$$

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O72b

(2) $\int \frac{x}{\sqrt{2-x}}dx$

[Sol] Let $\sqrt{2-x}=t$. Since $2-x=t^2$, $x=2-t^2$

$$\therefore\ dx=-2t\,dt$$

$$\begin{aligned}\therefore\ \int \frac{x}{\sqrt{2-x}}dx&=\int \frac{2-t^2}{t}\cdot(-2t)dt\\&=2\int(t^2-2)dt\\&=2\left(\frac{1}{3}t^3-2t\right)+C\\&=\frac{2}{3}t(t^2-6)+C\\&=-\frac{2}{3}(x+4)\sqrt{2-x}+C\end{aligned}$$

(3) $\int \frac{x^2}{\sqrt{x+2}}dx$

[Sol] Let $\sqrt{x+2}=t$. Since $x+2=t^2$, $x=t^2-2$

$$\therefore\ dx=2t\,dt$$

$$\begin{aligned}\therefore\ \int \frac{x^2}{\sqrt{x+2}}dx&=\int \frac{(t^2-2)^2}{t}\cdot 2t\,dt\\&=2\int(t^4-4t^2+4)dt\\&=2\left(\frac{1}{5}t^5-\frac{4}{3}t^3+4t\right)+C\\&=\frac{2}{15}t(3t^4-20t^2+60)+C\\&=\frac{2}{15}(3x^2-8x+32)\sqrt{x+2}+C\end{aligned}$$

Integration by Substitution

Name

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(mistakes) 0	—	—	1	2~

1. Find the following indefinite integrals using Integration by Substitution.

(1) $\int x\sqrt{2x-1}\,dx$

[Sol] Let $2x-1=t$. Since $x=\frac{t+1}{2}$, $dx=\frac{1}{2}dt$

$$\therefore \int x\sqrt{2x-1}\,dx=\int \frac{t+1}{2}\cdot\sqrt{t}\cdot\frac{1}{2}dt$$
$$=\frac{1}{4}\int\left(t^{\frac{3}{2}}+t^{\frac{1}{2}}\right)dt$$
$$=\frac{1}{4}\left(\frac{2}{5}t^{\frac{5}{2}}+\frac{2}{3}t^{\frac{3}{2}}\right)+C$$
$$=\frac{1}{30}t^{\frac{3}{2}}(3t+5)+C$$
$$=\mathbf{\frac{1}{15}(3x+1)(2x-1)\sqrt{2x-1}+C}$$

(2) $\int \frac{3x-1}{\sqrt{x+1}}dx$

[Sol] Let $x+1=t$. Since $x=t-1$, $dx=dt$

$$\therefore \int \frac{3x-1}{\sqrt{x+1}}dx=\int\frac{3t-4}{\sqrt{t}}dt$$
$$=\int\left(3t^{\frac{1}{2}}-4t^{-\frac{1}{2}}\right)dt$$
$$=2t^{\frac{3}{2}}-8t^{\frac{1}{2}}+C$$
$$=2t^{\frac{1}{2}}(t-4)+C$$
$$=\mathbf{2(x-3)\sqrt{x+1}+C}$$

Alternative Solution

Let $\sqrt{x+1}=t$. Since $x+1=t^2$, $x=t^2-1$ $\therefore dx=2t\,dt$

$$\therefore \int\frac{3x-1}{\sqrt{x+1}}dx=\int\frac{3t^2-4}{t}\cdot 2t\,dt=2\int(3t^2-4)dt=2(t^3-4t)+C$$
$$=2t(t^2-4)+C=\mathbf{2(x-3)\sqrt{x+1}+C}$$

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O73b

2. Find the indefinite integral $\int(2x+3)\sqrt{2x+1}\,dx$ using two different methods of Integration by Substitution.

(1) Let $\sqrt{2x+1}=t$.

[Sol] Since $2x+1=t^2$, $x=\frac{t^2-1}{2}$

$\therefore dx=t\,dt$

$$\therefore \int(2x+3)\sqrt{2x+1}\,dx=\int(t^2+2)t\cdot t\,dt$$

$$=\int(t^4+2t^2)dt$$

$$=\frac{1}{5}t^5+\frac{2}{3}t^3+C$$

$$=\frac{1}{15}t^3(3t^2+10)+C$$

$$=\frac{1}{15}(6x+13)(2x+1)\sqrt{2x+1}+C$$

(2) Let $2x+1=t$.

[Sol] Since $x=\frac{t-1}{2}$, $dx=\frac{1}{2}dt$

$$\therefore \int(2x+3)\sqrt{2x+1}\,dx=\int(t+2)\sqrt{t}\cdot\frac{1}{2}dt$$

$$=\frac{1}{2}\int\left(t^{\frac{3}{2}}+2t^{\frac{1}{2}}\right)dt$$

$$=\frac{1}{2}\left(\frac{2}{5}t^{\frac{5}{2}}+\frac{4}{3}t^{\frac{3}{2}}\right)+C$$

$$=\frac{1}{15}t^{\frac{3}{2}}(3t+10)+C$$

$$=\frac{1}{15}(6x+13)(2x+1)\sqrt{2x+1}+C$$

For question 2, do (1) and (2) have the same answer?

Integration by Substitution

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For Integration by Substitution I in O71, if the LHS and the RHS are switched and t and x are replaced by x and u respectively, then the following formula is obtained.

Integration by Substitution II

$$\int f(g(x))g'(x)dx=\int f(u)du \qquad (g(x)=u)$$

Find the following indefinite integrals using Integration by Substitution.

Ex. $\int (x^3+1)^2x^2dx$

[Sol] Let $x^3+1=u$. $3x^2dx=du$

$$\therefore \int (x^3+1)^2x^2dx=\int u^2\cdot\frac{1}{3}du \quad \longleftarrow \quad x^2dx=\frac{1}{3}du$$

$$=\frac{1}{3}\int u^2du$$

$$=\frac{1}{9}u^3+C$$

$$=\frac{1}{9}(x^3+1)^3+C$$

(1) $\int x(x^2+1)^3dx$

[Sol] Let $x^2+1=u$. $2xdx=du$

$$\therefore \int x(x^2+1)^3dx=\int u^3\cdot\frac{1}{2}du \quad \longleftarrow \quad xdx=\frac{1}{2}du$$

$$=\frac{1}{2}\int u^3du$$

$$=\frac{1}{8}u^4+C$$

$$=\frac{1}{8}(x^2+1)^4+C$$

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074b

(2) $\int \frac{x}{x^2-1}dx$

[Sol] Let $x^2-1=u$. $2x\,dx=du$

$$\begin{aligned}\therefore \int \frac{x}{x^2-1}dx &= \int \frac{1}{u}\cdot\frac{1}{2}du \quad \Longleftarrow \quad \boxed{x\,dx=\frac{1}{2}du}\\ &= \frac{1}{2}\int\frac{du}{u}\\ &= \frac{1}{2}\ln|u|+C\\ &= \frac{1}{2}\ln|x^2-1|+C\end{aligned}$$

(3) $\int x\sqrt{x^2+1}\,dx$

[Sol] Let $\sqrt{x^2+1}=u$. Since $x^2+1=u^2$, $2x\,dx=2u\,du$

$\therefore\ x\,dx=u\,du$

$$\begin{aligned}\therefore \int x\sqrt{x^2+1}\,dx &= \int u\cdot u\,du\\ &= \int u^2du\\ &= \frac{1}{3}u^3+C\\ &= \frac{1}{3}(x^2+1)\sqrt{x^2+1}+C\end{aligned}$$

Alternative Solution

Let $x^2+1=u$. $2x\,dx=du$

$$\begin{aligned}\therefore \int x\sqrt{x^2+1}\,dx &= \int\sqrt{u}\cdot\frac{1}{2}du\\ &= \frac{1}{2}\int u^{\frac{1}{2}}du\\ &= \frac{1}{3}u^{\frac{3}{2}}+C\\ &= \frac{1}{3}(x^2+1)\sqrt{x^2+1}+C\end{aligned}$$

(4) $\int \frac{x^3}{\sqrt{x^2+1}}dx$

[Sol] Let $\sqrt{x^2+1}=u$. Since $x^2+1=u^2$, $2x\,dx=2u\,du$

$\therefore\ x\,dx=u\,du$

$$\begin{aligned}\therefore \int \frac{x^3}{\sqrt{x^2+1}}dx &= \int\frac{u^2-1}{u}\cdot u\,du\\ &= \int(u^2-1)du\\ &= \frac{1}{3}u^3-u+C\\ &= \frac{1}{3}u(u^2-3)+C\\ &= \frac{1}{3}(x^2-2)\sqrt{x^2+1}+C\end{aligned}$$

Alternative Solution

Let $x^2+1=u$. $2x\,dx=du$

$$\begin{aligned}\therefore \int \frac{x^3}{\sqrt{x^2+1}}dx &= \int\frac{u-1}{\sqrt{u}}\cdot\frac{1}{2}du\\ &= \frac{1}{2}\int\left(u^{\frac{1}{2}}-u^{-\frac{1}{2}}\right)du\\ &= \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}}-2u^{\frac{1}{2}}\right)+C\\ &= \frac{1}{3}u^{\frac{1}{2}}(u-3)+C\\ &= \frac{1}{3}(x^2-2)\sqrt{x^2+1}+C\end{aligned}$$

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Find the following indefinite integrals using Integration by Substitution.

Ex. $\displaystyle\int \frac{\ln x}{x}dx$

[Sol] Let $\ln x = u$. $\dfrac{1}{x}dx = du$

$$\therefore \int \frac{\ln x}{x}dx = \int u\,du$$
$$= \frac{1}{2}u^2 + C$$
$$= \frac{1}{2}(\ln x)^2 + C$$

(1) $\displaystyle\int \frac{dx}{x(\ln x)^2}$

[Sol] Let $\ln x = u$. $\dfrac{1}{x}dx = du$

$$\therefore \int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2}$$
$$= -\frac{1}{u} + C$$
$$= -\frac{1}{\ln x} + C$$

(2) $\displaystyle\int \frac{\sqrt{\ln x}}{x}dx$

[Sol] Let $\ln x = u$. $\dfrac{1}{x}dx = du$

$$\therefore \int \frac{\sqrt{\ln x}}{x}dx = \int \sqrt{u}\,du$$
$$= \int u^{\frac{1}{2}}du$$
$$= \frac{2}{3}u^{\frac{3}{2}} + C$$
$$= \frac{2}{3}\ln x\sqrt{\ln x} + C$$

Alternative Solution

Let $\sqrt{\ln x} = u$.

Since $\ln x = u^2$, $\dfrac{1}{x}dx = 2u\,du$

$$\therefore \int \frac{\sqrt{\ln x}}{x}dx = \int u \cdot 2u\,du$$
$$= 2\int u^2 du$$
$$= \frac{2}{3}u^3 + C$$
$$= \frac{2}{3}\ln x\sqrt{\ln x} + C$$

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075b

Ex.

$$\int \frac{e^x}{e^x+1}dx$$

[Sol] Let $e^x=u$. $e^x dx = du$

$$\therefore \int \frac{e^x}{e^x+1}dx = \int \frac{du}{u+1}$$
$$= \ln|u+1| + C$$
$$= \ln|e^x+1| + C$$
$$= \ln(e^x+1) + C$$

Since $e^x+1>0$, $|e^x+1|=e^x+1$

(3) $\int \frac{e^x}{(e^x+1)^2}dx$

[Sol] Let $e^x=u$. $e^x dx = du$

$$\therefore \int \frac{e^x}{(e^x+1)^2}dx = \int \frac{du}{(u+1)^2}$$
$$= -\frac{1}{u+1} + C$$
$$= -\frac{1}{e^x+1} + C$$

Alternative Solution

Let $e^x+1=u$. $e^x dx = du$

$$\therefore \int \frac{e^x}{(e^x+1)^2}dx = \int \frac{du}{u^2}$$
$$= -\frac{1}{u} + C$$
$$= -\frac{1}{e^x+1} + C$$

(4) $\int \frac{dx}{e^x - e^{-x}}$

[Sol] Let $e^x=u$. $e^x dx = du$

$$\therefore \int \frac{dx}{e^x-e^{-x}} = \int \frac{\boxed{e^x}}{e^{2x}-1}dx$$

Multiplying the numerator and the denominator by e^x

$e^{2x}=(e^x)^2=u^2$

$$= \int \frac{\boxed{1}}{u^2-1}du$$

$\frac{1}{u^2-1}$
$= \frac{1}{(u-1)(u+1)}$
$= \frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u+1}\right)$

$$= \frac{1}{2}\int \left(\frac{\boxed{1}}{u-1} - \frac{\boxed{1}}{u+1}\right)du$$
$$= \frac{1}{2}(\ln|u-1| - \ln|u+1|) + C$$
$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C$$
$$= \frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right| + C = \frac{1}{2}\ln\frac{|e^x-1|}{e^x+1} + C$$

Since $e^x+1>0$, $|e^x+1|=e^x+1$

Integration by Substitution

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Find the following indefinite integrals using Integration by Substitution.

Ex. $\int \cos^3 x \sin x \, dx$

[Sol] Let $\cos x = u$. $-\sin x \, dx = du$

$$\therefore \int \cos^3 x \sin x \, dx = \int u^3 \cdot (-1) \, du \quad \Leftarrow \boxed{\sin x \, dx = (-1) du}$$
$$= -\int u^3 \, du$$
$$= -\frac{1}{4} u^4 + C$$
$$= -\frac{1}{4} \cos^4 x + C$$

(1) $\int \cos^5 x \sin x \, dx$

[Sol] Let $\cos x = u$. $-\sin x \, dx = du$

$$\therefore \int \cos^5 x \sin x \, dx = \int u^5 \cdot (-1) \, du$$
$$= -\int u^5 \, du$$
$$= -\frac{1}{6} u^6 + C$$
$$= -\frac{1}{6} \cos^6 x + C$$

(2) $\int \sin^2 x \cos x \, dx$

[Sol] Let $\sin x = u$. $\cos x \, dx = du$

$$\therefore \int \sin^2 x \cos x \, dx = \int u^2 \, du$$
$$= \frac{1}{3} u^3 + C$$
$$= \frac{1}{3} \sin^3 x + C$$

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O76b

(3) $\int \tan x\,dx$

[Sol] Let $\cos x = u$. $-\sin x\,dx = du$

$$\therefore \int \tan x\,dx = \int \frac{\sin x}{\cos x}dx$$
$$= \int \frac{1}{u}\cdot(-1)du$$
$$= -\int \frac{du}{u}$$
$$= -\ln|u| + C$$
$$= -\ln|\cos x| + C$$

(4) $\int \cos^3 x\,dx$

[Sol] $\int \cos^3 x\,dx = \int (1-\sin^2 x)\cos x\,dx$ ⟵ $\cos^3 x = \cos^2 x \cdot \cos x$ $= (1-\sin^2 x)\cos x$

Let $\sin x = u$. $\cos x\,dx = du$

$$\therefore \int \cos^3 x\,dx = \int (1-u^2)du$$
$$= u - \frac{1}{3}u^3 + C$$
$$= \sin x - \frac{1}{3}\sin^3 x + C \quad \left[= \frac{1}{3}\sin x(3-\sin^2 x) + C\right]$$

(5) $\int \tan^3 x\,dx$

[Sol] $\int \tan^3 x\,dx = \int \frac{\boxed{\sin^3 x}}{\cos^3 x}dx = \int \frac{(1-\cos^2 x)\sin x}{\cos^3 x}dx$

Let $\cos x = u$. $-\sin x\,dx = du$

$$\therefore \int \tan^3 x\,dx = \int \frac{1-u^2}{u^3}\cdot(-1)du$$
$$= \int \left(\frac{1}{u} - \frac{1}{u^3}\right)du$$
$$= \ln|u| + \frac{1}{2u^2} + C$$
$$= \ln|\cos x| + \frac{1}{2\cos^2 x} + C$$

O77a

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Integration by Substitution

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Find the following indefinite integrals using Integration by Substitution.

(1) $\int \frac{dx}{1-\sin x}$

[Sol] $\int \frac{dx}{1-\sin x} = \int \frac{1+\sin x}{1-\sin^2 x}\,dx$ ← Multiplying the numerator and the denominator by $1+\sin x$

$= \int \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right) dx$ ← $1-\sin^2 x = \cos^2 x$

Since $\int \frac{1}{\cos^2 x}\,dx = \boxed{\tan x} + C_1$, find $\int \frac{\sin x}{\cos^2 x}\,dx$.

Let $\cos x = u$. $-\sin x\,dx = du$

$\therefore \int \frac{\sin x}{\cos^2 x}\,dx = \int \frac{1}{u^2}\cdot(-1)\,du = -\int \frac{du}{u^2}$

$= \frac{1}{u} + C_2 = \frac{1}{\cos x} + C_2$

$\therefore \int \frac{dx}{1-\sin x} = \tan x + \frac{1}{\cos x} + C$ ← The constants of integration are collectively expressed as C.

(2) $\int \frac{dx}{1-\cos x}$

[Sol] $\int \frac{dx}{1-\cos x} = \int \frac{1+\cos x}{1-\cos^2 x}\,dx$

$= \int \left(\frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x}\right) dx$

Since $\int \frac{1}{\sin^2 x}\,dx = -\frac{1}{\tan x} + C_1$, find $\int \frac{\cos x}{\sin^2 x}\,dx$.

Let $\sin x = u$. $\cos x\,dx = du$

$\therefore \int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{du}{u^2} = -\frac{1}{u} + C_2$

$= -\frac{1}{\sin x} + C_2$

$\therefore \int \frac{dx}{1-\cos x} = -\frac{1}{\tan x} - \frac{1}{\sin x} + C$

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O77b

(3) $\int \frac{dx}{\sin x}$

[Sol] $\int \frac{dx}{\sin x} = \int \frac{\boxed{\sin x}}{\sin^2 x} dx = \int \frac{\boxed{\sin x}}{1-\cos^2 x} dx$

Let $\cos x = u$. $-\sin x\, dx = du$

$$\therefore \int \frac{dx}{\sin x} = \int \frac{\boxed{1}}{1-u^2} \cdot (-1) du$$

$$= \int \frac{\boxed{1}}{u^2-1} du$$

$$= \frac{1}{2} \int \left(\frac{\boxed{1}}{u-1} - \frac{\boxed{1}}{u+1} \right) du$$

$$= \frac{1}{2} (\ln|u-1| - \ln|u+1|) + C$$

$$= \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| + C$$

$$= \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + C = \frac{1}{2} \ln \frac{1-\cos x}{1+\cos x} + C$$

$$\frac{1}{u^2-1} = \frac{1}{(u-1)(u+1)} = \frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u+1} \right)$$

From the integrand,
$-1 < \cos x < 1$
$\therefore \quad 0 < \cos x + 1 < 2$
$-2 < \cos x - 1 < 0$
$\therefore \ |\cos x + 1| = 1 + \cos x$
$|\cos x - 1| = 1 - \cos x$

(4) $\int \frac{dx}{\cos x}$

[Sol] $\int \frac{dx}{\cos x} = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{\cos x}{1-\sin^2 x} dx$

Let $\sin x = u$. $\cos x\, dx = du$

$$\frac{1}{1-u^2} = -\frac{1}{(u-1)(u+1)} = -\frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u+1} \right)$$

$$\therefore \int \frac{dx}{\cos x} = \int \frac{1}{1-u^2} du$$

$$= -\frac{1}{2} \int \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du$$

$$= -\frac{1}{2} (\ln|u-1| - \ln|u+1|) + C$$

$$= \frac{1}{2} \ln \left| \frac{u+1}{u-1} \right| + C$$

$$= \frac{1}{2} \ln \left| \frac{\sin x + 1}{\sin x - 1} \right| + C = \frac{1}{2} \ln \frac{1+\sin x}{1-\sin x} + C$$

$$\left[= -\frac{1}{2} \ln \frac{1-\sin x}{1+\sin x} + C \right]$$

From the integrand,
$-1 < \sin x < 1$
$\therefore \ -2 < \sin x - 1 < 0$
$0 < \sin x + 1 < 2$
$\therefore \ |\sin x - 1| = 1 - \sin x$
$|\sin x + 1| = 1 + \sin x$

Integration by Substitution

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For Integration by Substitution II in O74, let $f(u)=\frac{1}{u}$ $\left(\text{i.e. } f(g(x))=\frac{1}{g(x)}\right)$.

$$\int \frac{g'(x)}{g(x)}dx=\int \frac{1}{u}du=\ln|u|+C$$

Therefore, the following formula is true.

Indefinite Integral of $\frac{g'(x)}{g(x)}$

$$\int \frac{g'(x)}{g(x)}dx=\ln|g(x)|+C$$

Find the following indefinite integrals using the formula above.

Ex.

[1] $\int \frac{2x}{x^2-3}dx=\int \frac{(x^2-3)'}{x^2-3}dx=\ln|x^2-3|+C$

[2] $\int \frac{2x}{x^2+3}dx=\int \frac{(x^2+3)'}{x^2+3}dx=\ln|x^2+3|+C=\ln(x^2+3)+C$

Since $x^2+3>0$, $|x^2+3|=x^2+3$

(1) $\int \frac{2x+1}{x^2+x-1}dx=\int \frac{(x^2+x-1)'}{x^2+x-1}dx=\ln|x^2+x-1|+C$

(2) $\int \frac{4x^3}{x^4+1}dx=\int \frac{(x^4+1)'}{x^4+1}dx=\ln|x^4+1|+C=\ln(x^4+1)+C$

(3) $\int \frac{x-1}{x^2-2x-3}dx=\int \frac{(x^2-2x-3)'}{x^2-2x-3}\cdot\frac{1}{2}dx=\frac{1}{2}\ln|x^2-2x-3|+C$

O78b

(4) $\int \frac{2x-4}{x^2-4x+5}dx = \int \frac{(x^2-4x+5)'}{x^2-4x+5}dx$

$= \ln|x^2-4x+5| + C$

$= \ln(x^2-4x+5) + C$

Since $x^2-4x+5=(x-2)^2+1>0$,
$|x^2-4x+5| = x^2-4x+5$

(5) $\int \frac{\sin x}{1-\cos x}dx = \int \frac{(1-\cos x)'}{1-\cos x}dx = \ln|1-\cos x| + C = \ln(1-\cos x) + C$

From the integrand, $\cos x \neq 1$
$\therefore\ 1-\cos x > 0$
$\therefore\ |1-\cos x| = 1-\cos x$

(6) $\int \frac{\sin x - \cos x}{\sin x + \cos x}dx = \int \frac{(\sin x + \cos x)'}{\sin x + \cos x}\cdot(-1)dx = -\ln|\sin x + \cos x| + C$

(7) $\int \frac{dx}{\tan x} = \int \frac{\cos x}{\sin x}dx = \int \frac{(\sin x)'}{\sin x}dx = \ln|\sin x| + C$

(8) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \int \frac{(e^x+e^{-x})'}{e^x+e^{-x}}dx = \ln|e^x+e^{-x}| + C = \ln(e^x+e^{-x}) + C$

Since $e^x > 0$ and $e^{-x} > 0$,
$|e^x+e^{-x}| = e^x+e^{-x}$

(9) $\int \frac{dx}{x\ln x} = \int \frac{(\ln x)'}{\ln x}dx = \ln|\ln x| + C$

Integration by Substitution

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1. Given that the function $F(x)$ satisfies $F'(x)=xe^{x^2}$ and $F(0)=0$, find $F(x)$.

[Sol] Since $F'(x)=xe^{x^2}$, $F(x)=\int xe^{x^2}dx$

Let $x^2=u$. $2x\,dx=du$

$$\therefore\ F(x)=\int e^u\cdot\frac{1}{2}du$$
$$=\frac{1}{2}\int e^u du$$
$$=\frac{1}{2}e^u+C$$
$$=\frac{1}{2}e^{x^2}+C$$

Since $F(0)=0$, $C=-\frac{1}{2}$

$$\therefore\ F(x)=\frac{1}{2}e^{x^2}-\frac{1}{2}\quad\left[=\frac{1}{2}(e^{x^2}-1)\right]$$

Alternative Solution
The answer can be derived as shown below.
Let $e^{x^2}=u$. $2xe^{x^2}dx=du$
Therefore,
$$F(x)=\int\frac{1}{2}du$$
$$=\frac{1}{2}u+C$$
$$=\frac{1}{2}e^{x^2}+C$$

2. Given that the function $F(x)$ satisfies $F'(x)=\frac{\sin x}{2+\cos x}$ and $F(0)=0$, find $F(x)$.

[Sol] Since $F'(x)=\frac{\sin x}{2+\cos x}$, $F(x)=\int\frac{\sin x}{2+\cos x}dx$

Let $\cos x=u$. $-\sin x\,dx=du$

$$\therefore\ F(x)=\int\frac{1}{2+u}\cdot(-1)du$$
$$=-\int\frac{du}{2+u}$$
$$=-\ln|2+u|+C$$
$$=-\ln|2+\cos x|+C$$
$$=-\ln(2+\cos x)+C$$

Since $-1\leqq\cos x\leqq1$, $|2+\cos x|=2+\cos x$

Since $F(0)=0$, $C=\ln 3$

$\ln M-\ln N=\ln\frac{M}{N}$

$$\therefore\ F(x)=-\ln(2+\cos x)+\ln 3=\ln\frac{3}{2+\cos x}$$

Alternative Solution
The answer can be derived as shown below.
$$F(x)=\int\frac{(2+\cos x)'}{2+\cos x}\cdot(-1)dx$$
$$=-\ln|2+\cos x|+C$$
$$=-\ln(2+\cos x)+C$$

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O79b

3. Find the indefinite integral $\int \frac{ds}{\sqrt{s^2-1}}$ by letting $s+\sqrt{s^2-1}=t$.

[Sol] Since $s+\sqrt{s^2-1}=t$, $s^2-1=(t-s)^2$

When $t=0$, $s^2-1=s^2$
There are no real numbers s that satisfy the above.
$\therefore\ t\neq 0$

$$\therefore\ s=\frac{t^2+1}{2t}=\frac{1}{2}\left(t+\frac{1}{t}\right)$$

$$ds=\frac{1}{2}\left(1-\frac{1}{t^2}\right)dt=\frac{t^2-1}{2t^2}dt$$

Also, $\sqrt{s^2-1}=t-s=\frac{1}{2}\left(t-\frac{1}{t}\right)=\frac{t^2-1}{2t}$

Since $s=\frac{1}{2}\left(t+\frac{1}{t}\right)$

$$\therefore\ \int \frac{ds}{\sqrt{s^2-1}}=\int \frac{2t}{t^2-1}\cdot\frac{t^2-1}{2t^2}dt$$

$$=\int \frac{dt}{t}$$

$$=\ln|t|+C$$

$$=\ln|s+\sqrt{s^2-1}|+C$$

Integration by Substitution

Name

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Time : to :

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(mistakes) 0	—	1	—	2~

Find the following indefinite integrals using Integration by Substitution.

(1) $\displaystyle\int \frac{x}{\sqrt{2x+1}}\,dx$ ➡ O72

[Sol] Let $\sqrt{2x+1}=t$. Since $2x+1=t^2$, $x=\dfrac{t^2-1}{2}$

$$\therefore\ dx=t\,dt$$

$$\begin{aligned}\therefore\ \int \frac{x}{\sqrt{2x+1}}\,dx&=\int \frac{1}{t}\cdot\frac{t^2-1}{2}\cdot t\,dt\\&=\frac{1}{2}\int (t^2-1)\,dt\\&=\frac{1}{2}\left(\frac{1}{3}t^3-t\right)+C\\&=\frac{1}{6}t(t^2-3)+C\\&=\boldsymbol{\frac{1}{3}(x-1)\sqrt{2x+1}+C}\end{aligned}$$

Alternative Solution

Let $2x+1=t$. Since $x=\dfrac{t-1}{2}$, $dx=\dfrac{1}{2}dt$

$$\therefore\ \int \frac{x}{\sqrt{2x+1}}\,dx=\int \frac{1}{\sqrt{t}}\cdot\frac{t-1}{2}\cdot\frac{1}{2}\,dt=\frac{1}{4}\int\left(t^{\frac{1}{2}}-t^{-\frac{1}{2}}\right)dt=\frac{1}{4}\left(\frac{2}{3}t^{\frac{3}{2}}-2t^{\frac{1}{2}}\right)+C$$

$$=\frac{1}{6}t^{\frac{1}{2}}(t-3)+C=\boldsymbol{\frac{1}{3}(x-1)\sqrt{2x+1}+C}$$

(2) $\displaystyle\int \frac{2x^2}{x^3+1}\,dx$ ➡ O74

[Sol] Let $x^3+1=u$. $3x^2\,dx=du$

$$\begin{aligned}\therefore\ \int \frac{2x^2}{x^3+1}\,dx&=\int \frac{2}{u}\cdot\frac{1}{3}\,du\\&=\frac{2}{3}\int\frac{du}{u}\\&=\frac{2}{3}\ln|u|+C\\&=\boldsymbol{\frac{2}{3}\ln|x^3+1|+C}\end{aligned}$$

Alternative Solution

$$\int \frac{2x^2}{x^3+1}\,dx=\int \frac{(x^3+1)'}{x^3+1}\cdot\frac{2}{3}\,dx=\boldsymbol{\frac{2}{3}\ln|x^3+1|+C}$$

O80b

(3) $\int \frac{(\ln x)^2}{x} dx$ ➡ O75

[Sol] Let $\ln x = u$. $\frac{1}{x} dx = du$

$$\therefore \int \frac{(\ln x)^2}{x} dx = \int u^2 du$$
$$= \frac{1}{3} u^3 + C$$
$$= \frac{1}{3} (\ln x)^3 + C$$

(4) $\int \frac{\cos x}{1+\sin x} dx$ ➡ O76

[Sol] Let $\sin x = u$. $\cos x\, dx = du$

$$\therefore \int \frac{\cos x}{1+\sin x} dx = \int \frac{du}{1+u}$$
$$= \ln|1+u| + C$$
$$= \ln|1+\sin x| + C$$
$$= \ln(1+\sin x) + C$$

[Alternative Solution

$$\int \frac{\cos x}{1+\sin x} dx = \int \frac{(1+\sin x)'}{1+\sin x} dx = \ln|1+\sin x| + C = \ln(1+\sin x) + C$$]

(5) $\int \sin^2 x \cos^3 x\, dx$ ➡ O76

[Sol] $\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x (1 - \sin^2 x) \cos x\, dx$

Let $\sin x = u$. $\cos x\, dx = du$

$$\therefore \int \sin^2 x \cos^3 x\, dx = \int u^2 (1-u^2) du$$
$$= \int (u^2 - u^4) du$$
$$= \frac{1}{3} u^3 - \frac{1}{5} u^5 + C$$
$$= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C$$
$$\left[= \frac{1}{15} \sin^3 x (5 - 3\sin^2 x) + C \right]$$

Integration by Parts

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The Product Rule

$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$ ← N144

can be rearranged into

$f(x)g'(x) = [f(x)g(x)]' - f'(x)g(x)$

If both sides are integrated, then the following formula is obtained.

Integration by Parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

This method is called ***Integration by Parts***.

Find the following indefinite integrals using Integration by Parts.

Ex.

$$\int x\cos x\,dx = \int x\,(\sin x)'\,dx \quad \leftarrow \int f(x)\,g'(x)\,dx$$

$$= x\sin x - \int (x)'\sin x\,dx \quad \leftarrow f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

$$= x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

(1) $\int x\sin x\,dx = \int x(-\cos x)'dx$

$$= -x\cos x - \int (x)'(-\cos x)dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

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081b

(2) $\int xe^x dx = \int x(e^x)' dx$

$= xe^x - \int (x)' e^x dx$

$= xe^x - \int e^x dx$

$= xe^x - e^x + C$

$\left[= (x-1)e^x + C\right]$

(3) $\int xe^{2x} dx = \int x\left(\frac{1}{2}e^{2x}\right)' dx$

$= \frac{1}{2}xe^{2x} - \int (x)' \cdot \frac{1}{2}e^{2x} dx$

$= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x} dx$

$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$

$\left[= \frac{1}{4}(2x-1)e^{2x} + C\right]$

(4) $\int x\cos 3x\, dx = \int x\left(\frac{1}{3}\sin 3x\right)' dx$

$= \frac{1}{3}x\sin 3x - \int (x)' \cdot \frac{1}{3}\sin 3x\, dx$

$= \frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x\, dx$

$= \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C$

$\left[= \frac{1}{9}(3x\sin 3x + \cos 3x) + C\right]$

(5) $\int x(x-1)^5 dx = \int x\left[\frac{1}{6}(x-1)^6\right]' dx$

$= \frac{1}{6}x(x-1)^6 - \int (x)' \cdot \frac{1}{6}(x-1)^6 dx$

$= \frac{1}{6}x(x-1)^6 - \frac{1}{6}\int (x-1)^6 dx$

$= \frac{1}{6}x(x-1)^6 - \frac{1}{42}(x-1)^7 + C$

$= \frac{1}{42}(6x+1)(x-1)^6 + C$

Integration by Parts

Name

Date / /

Time : to :

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1. Find the following indefinite integrals using Integration by Parts.

Ex.

$$\int x\ln x\,dx=\int\left(\frac{1}{2}x^2\right)'\ln x\,dx$$
$$=\frac{1}{2}x^2\ln x-\int\frac{1}{2}x^2(\ln x)'dx$$
$$(\ln x)'=\frac{1}{x}$$
$$=\frac{1}{2}x^2\ln x-\frac{1}{2}\int x\,dx$$
$$=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$$

(1) $$\int x^4\ln x\,dx=\int\left(\frac{1}{5}x^5\right)'\ln x\,dx$$
$$=\frac{1}{5}x^5\ln x-\int\frac{1}{5}x^5(\ln x)'dx$$
$$=\frac{1}{5}x^5\ln x-\frac{1}{5}\int x^4dx$$
$$=\frac{1}{5}x^5\ln x-\frac{1}{25}x^5+C$$
$$\left[=\frac{1}{25}x^5(5\ln x-1)+C\right]$$

(2) $$\int\sqrt{x}\ln x\,dx=\int\left(\frac{2}{3}x^{\frac{3}{2}}\right)'\ln x\,dx$$
$$=\frac{2}{3}x^{\frac{3}{2}}\ln x-\int\frac{2}{3}x^{\frac{3}{2}}(\ln x)'dx$$
$$=\frac{2}{3}x\sqrt{x}\ln x-\frac{2}{3}\int x^{\frac{1}{2}}dx$$
$$=\frac{2}{3}x\sqrt{x}\ln x-\frac{4}{9}x^{\frac{3}{2}}+C$$
$$=\frac{2}{3}x\sqrt{x}\ln x-\frac{4}{9}x\sqrt{x}+C$$
$$\left[=\frac{2}{9}x\sqrt{x}(3\ln x-2)+C\right]$$

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O82b

(3) $$\int x\ln(x^2+1)dx=\int\left(\frac{1}{2}x^2\right)'\ln(x^2+1)dx$$
$$=\frac{1}{2}x^2\ln(x^2+1)-\int\frac{1}{2}x^2[\ln(x^2+1)]'dx$$
$$=\frac{1}{2}x^2\ln(x^2+1)-\int\frac{x^3}{x^2+1}dx$$
$$=\frac{1}{2}x^2\ln(x^2+1)-\int\left(x-\frac{x}{x^2+1}\right)dx$$

$$\begin{array}{r} x \\ x^2+1\overline{\smash{)}x^3} \\ \underline{x^3+x} \\ -x \end{array}$$

$$=\frac{1}{2}x^2\ln(x^2+1)-\int\left[x-\frac{(x^2+1)'}{x^2+1}\cdot\frac{1}{2}\right]dx$$
$$=\frac{1}{2}x^2\ln(x^2+1)-\left[\frac{1}{2}x^2-\frac{1}{2}\ln(x^2+1)\right]+C$$
$$=\mathbf{\frac{1}{2}(x^2+1)\ln(x^2+1)-\frac{1}{2}x^2+C}$$

2. Find the indefinite integral $\int e^x\ln(e^x+1)dx$ using two different methods of Integration by Parts.

(1) $$\int e^x\ln(e^x+1)dx=\int(e^x)'\ln(e^x+1)dx$$
$$=e^x\ln(e^x+1)-\int e^x[\ln(e^x+1)]'dx$$
$$=e^x\ln(e^x+1)-\int\frac{e^{2x}}{e^x+1}dx$$
$$=e^x\ln(e^x+1)-\int\left(e^x-\frac{e^x}{e^x+1}\right)dx$$

$$\frac{e^{2x}}{e^x+1}=\frac{e^x(e^x+1)-e^x}{e^x+1}=e^x-\frac{e^x}{e^x+1}$$

$$=e^x\ln(e^x+1)-\int\left[e^x-\frac{(e^x+1)'}{e^x+1}\right]dx$$
$$=e^x\ln(e^x+1)-[e^x-\ln(e^x+1)]+C$$
$$=\mathbf{(e^x+1)\ln(e^x+1)-e^x+C}$$

(2) $$\int e^x\ln(e^x+1)dx=\int(e^x+1)'\ln(e^x+1)dx$$
$$=(e^x+1)\ln(e^x+1)-\int(e^x+1)[\ln(e^x+1)]'dx$$
$$=(e^x+1)\ln(e^x+1)-\int e^x dx$$
$$=\mathbf{(e^x+1)\ln(e^x+1)-e^x+C}$$

For question 2, do (1) and (2) have the same answer?

Integration by Parts

Name

Date / /

Time : to :

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1. Find the following indefinite integrals using Integration by Parts.

Ex. $\int \ln(x+1)dx = \int (x)'\ln(x+1)dx$ ⟵ $\int \ln(x+1)dx = \int 1\cdot\ln(x+1)dx$

$=x\ln(x+1) - \int x[\ln(x+1)]'dx$

$=x\ln(x+1) - \int \frac{x}{x+1}dx$

$=x\ln(x+1) - \int \left(1 - \frac{1}{x+1}\right)dx$

$$x+1\overline{)x}\quad 1,\quad \frac{x+1}{-1}$$

$=x\ln(x+1) - [x - \ln(x+1)] + C$

$=(x+1)\ln(x+1) - x + C$

(1) $\int \ln(x-2)dx = \int (x)'\ln(x-2)dx$

$=x\ln(x-2) - \int x[\ln(x-2)]'dx$

$=x\ln(x-2) - \int \frac{x}{x-2}dx$

$=x\ln(x-2) - \int \left(1 + \frac{2}{x-2}\right)dx$

$=x\ln(x-2) - [x + 2\ln(x-2)] + C$

$=(x-2)\ln(x-2) - x + C$

(2) $\int \ln x\,dx = \int (x)'\ln x\,dx$

$=x\ln x - \int x(\ln x)'dx$

$=x\ln x - \int dx$

$=x\ln x - x + C$

When integrating logarithmic functions, the method shown in **Ex.** is often used.

O83b

(3) $\int \ln(1-x)dx = \int (x)'\ln(1-x)dx$

$$= x\ln(1-x) - \int x[\ln(1-x)]'dx$$

$$= x\ln(1-x) + \int \frac{x}{1-x}dx$$

$$= x\ln(1-x) + \int \left(-1+\frac{1}{1-x}\right)dx$$

$$= x\ln(1-x) + [-x-\ln(1-x)] + C$$

$$= \mathbf{(x-1)\ln(1-x)-x+C}$$

$$\left[= -(1-x)\ln(1-x)-x+C\right]$$

2. Find the indefinite integral $\int \ln(2x+1)dx$ using two different methods of Integration by Parts.

(1) $\int \ln(2x+1)dx = \int (x)'\ln(2x+1)dx$

$$= x\ln(2x+1) - \int x[\ln(2x+1)]'dx$$

$$= x\ln(2x+1) - \int \frac{2x}{2x+1}dx$$

$$= x\ln(2x+1) - \int \left(1-\frac{1}{2x+1}\right)dx$$

$$= x\ln(2x+1) - \left[x-\frac{1}{2}\ln(2x+1)\right] + C$$

$$= \mathbf{\frac{1}{2}(2x+1)\ln(2x+1)-x+C}$$

(2) $\int \ln(2x+1)dx = \int \left[\frac{1}{2}(2x+1)\right]'\ln(2x+1)dx$

$$= \frac{1}{2}(2x+1)\ln(2x+1) - \int \frac{1}{2}(2x+1)[\ln(2x+1)]'dx$$

$$= \frac{1}{2}(2x+1)\ln(2x+1) - \int dx$$

$$= \mathbf{\frac{1}{2}(2x+1)\ln(2x+1)-x+C}$$

For question 2, do (1) and (2) have the same answer?

Integration by Parts

Name

Date / /

Time : to :

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Find the following indefinite integrals using Integration by Parts.

Ex.

$$\int x^2\sin x\,dx=\int x^2(-\cos x)'dx$$
$$=-x^2\cos x-\int (x^2)'(-\cos x)dx$$
$$=-x^2\cos x+2\int x\cos x\,dx$$
$$=-x^2\cos x+2\boxed{\int x(\sin x)'dx}$$
$$=-x^2\cos x+2\boxed{\left[x\sin x-\int (x)'\sin x\,dx\right]}$$
$$=-x^2\cos x+2x\sin x-2\int \sin x\,dx$$
$$=-x^2\cos x+2x\sin x+2\cos x+C$$

Using Integration by Parts again

(1) $$\int x^2\cos x\,dx=\int x^2(\sin x)'dx$$
$$=x^2\sin x-\int (x^2)'\sin x\,dx$$
$$=x^2\sin x-2\int x\sin x\,dx$$
$$=x^2\sin x-2\int x(-\cos x)'dx$$
$$=x^2\sin x-2\left[-x\cos x-\int (x)'(-\cos x)dx\right]$$
$$=x^2\sin x+2x\cos x-2\int \cos x\,dx$$
$$=\mathbf{x^2\sin x+2x\cos x-2\sin x+C}$$

(2) $$\int x^2e^x dx=\int x^2(e^x)'dx$$
$$=x^2e^x-\int (x^2)'e^x dx$$
$$=x^2e^x-2\int xe^x dx$$
$$=x^2e^x-2\int x(e^x)'dx$$
$$=x^2e^x-2\left[xe^x-\int (x)'e^x dx\right]$$
$$=x^2e^x-2xe^x+2\int e^x dx$$
$$=\mathbf{x^2e^x-2xe^x+2e^x+C}$$
$$\left[=\mathbf{(x^2-2x+2)e^x+C}\right]$$

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084b

(3) $\int(\ln x)^2dx=\int(x)'(\ln x)^2dx$

$=x(\ln x)^2-\int x[(\ln x)^2]'dx$

$=x(\ln x)^2-2\int\ln x\,dx$

$=x(\ln x)^2-2\int(x)'\ln x\,dx$

$=x(\ln x)^2-2\left[x\ln x-\int x(\ln x)'dx\right]$

$=x(\ln x)^2-2x\ln x+2\int dx$

$=x(\ln x)^2-2x\ln x+2x+C$

$[=x[(\ln x)^2-2\ln x+2]+C]$

$\int x[(\ln x)^2]'dx$

$=\int x\cdot 2\ln x\cdot\frac{1}{x}dx$

$=2\int\ln x\,dx$

(4) $\int x^2(\ln x)^2dx=\int\left(\frac{1}{3}x^3\right)'(\ln x)^2dx$

$=\frac{1}{3}x^3(\ln x)^2-\int\frac{1}{3}x^3[(\ln x)^2]'dx$

$=\frac{1}{3}x^3(\ln x)^2-\frac{2}{3}\int x^2\ln x\,dx$

$=\frac{1}{3}x^3(\ln x)^2-\frac{2}{3}\int\left(\frac{1}{3}x^3\right)'\ln x\,dx$

$=\frac{1}{3}x^3(\ln x)^2-\frac{2}{3}\left[\frac{1}{3}x^3\ln x-\int\frac{1}{3}x^3(\ln x)'dx\right]$

$=\frac{1}{3}x^3(\ln x)^2-\frac{2}{9}x^3\ln x+\frac{2}{9}\int x^2dx$

$=\frac{1}{3}x^3(\ln x)^2-\frac{2}{9}x^3\ln x+\frac{2}{27}x^3+C$

$\left[=\frac{1}{27}x^3[9(\ln x)^2-6\ln x+2]+C\right]$

O85a KUMON® O 85

Integration by Parts

Name

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Time : to :

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Find the following indefinite integrals using Integration by Parts.

(1) $\int x^2e^{1-x}dx=\int x^2(-e^{1-x})'dx$

$=-x^2e^{1-x}-\int(x^2)'(-e^{1-x})dx$

$=-x^2e^{1-x}+2\int xe^{1-x}dx$

$=-x^2e^{1-x}+2\int x(-e^{1-x})'dx$

$=-x^2e^{1-x}+2\left[-xe^{1-x}-\int(x)'(-e^{1-x})dx\right]$

$=-x^2e^{1-x}-2xe^{1-x}+2\int e^{1-x}dx$

$=-x^2e^{1-x}-2xe^{1-x}-2e^{1-x}+C$

$\left[=-(x^2+2x+2)e^{1-x}+C\right]$

(2) $\int(x^2-x)e^{-x}dx=\int(x^2-x)(-e^{-x})'dx$

$=-(x^2-x)e^{-x}-\int(x^2-x)'(-e^{-x})dx$

$=-(x^2-x)e^{-x}+\int(2x-1)e^{-x}dx$

$=-(x^2-x)e^{-x}+\int(2x-1)(-e^{-x})'dx$

$=-(x^2-x)e^{-x}+\left[-(2x-1)e^{-x}-\int(2x-1)'(-e^{-x})dx\right]$

$=-(x^2+x-1)e^{-x}+2\int e^{-x}dx$

$=-(x^2+x-1)e^{-x}-2e^{-x}+C$

$=-(x^2+x+1)e^{-x}+C$

$\left[=-x^2e^{-x}-xe^{-x}-e^{-x}+C\right]$

O85b

(3) $\int x^2e^{2x}dx=\int x^2\left(\frac{1}{2}e^{2x}\right)'dx$

$=\frac{1}{2}x^2e^{2x}-\int(x^2)'\cdot\frac{1}{2}e^{2x}dx$

$=\frac{1}{2}x^2e^{2x}-\int xe^{2x}dx$

$=\frac{1}{2}x^2e^{2x}-\int x\left(\frac{1}{2}e^{2x}\right)'dx$

$=\frac{1}{2}x^2e^{2x}-\left[\frac{1}{2}xe^{2x}-\int(x)'\cdot\frac{1}{2}e^{2x}dx\right]$

$=\frac{1}{2}x^2e^{2x}-\frac{1}{2}xe^{2x}+\frac{1}{2}\int e^{2x}dx$

$=\frac{1}{2}x^2e^{2x}-\frac{1}{2}xe^{2x}+\frac{1}{4}e^{2x}+C$

$\left[=\frac{1}{4}(2x^2-2x+1)e^{2x}+C\right]$

(4) $\int x^3e^xdx=\int x^3(e^x)'dx$

$=x^3e^x-\int(x^3)'e^xdx$

$=x^3e^x-3\int x^2e^xdx$

$=x^3e^x-3\int x^2(e^x)'dx$

$=x^3e^x-3\left[x^2e^x-\int(x^2)'e^xdx\right]$

$=x^3e^x-3x^2e^x+6\int xe^xdx$

$=x^3e^x-3x^2e^x+6\int x(e^x)'dx$

$=x^3e^x-3x^2e^x+6\left[xe^x-\int(x)'e^xdx\right]$ Using Integration by Parts again

$=x^3e^x-3x^2e^x+6xe^x-6\int e^xdx$

$=x^3e^x-3x^2e^x+6xe^x-6e^x+C$

$\left[=(x^3-3x^2+6x-6)e^x+C\right]$

Integration by Parts

Name

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Find the following indefinite integrals using Integration by Parts.

Ex.

$$\int e^x \sin x\,dx = \int (e^x)'\sin x\,dx$$
$$= e^x\sin x - \int e^x(\sin x)'dx$$
$$= e^x\sin x - \int e^x\cos x\,dx$$
$$= e^x\sin x - \int (e^x)'\cos x\,dx$$
$$= e^x\sin x - \left[e^x\cos x - \int e^x(\cos x)'dx\right]$$
$$= e^x\sin x - e^x\cos x - \int e^x\sin x\,dx$$

Same as the given expression

$$\therefore\ 2\int e^x\sin x\,dx = e^x(\sin x - \cos x) + C_1$$

Considering the constant of integration,

$$\int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C$$

The constant of integration for the indefinite integral to be found is expressed as $C\left(=\frac{1}{2}C_1\right)$.

(1) $\int e^x\cos x\,dx = \int (e^x)'\cos x\,dx$

$$= e^x\cos x - \int e^x(\cos x)'dx$$
$$= e^x\cos x + \int e^x\sin x\,dx$$
$$= e^x\cos x + \int (e^x)'\sin x\,dx$$
$$= e^x\cos x + \left[e^x\sin x - \int e^x(\sin x)'dx\right]$$
$$= e^x\cos x + e^x\sin x - \int e^x\cos x\,dx$$

$$\therefore\ 2\int e^x\cos x\,dx = e^x(\cos x + \sin x) + C_1$$

Considering the constant of integration,

$$\int e^x\cos x\,dx = \frac{1}{2}e^x(\cos x + \sin x) + C$$

O86b

(2) $\int \frac{\ln x}{x} dx = \int \ln x (\ln x)' dx$

$= (\ln x)^2 - \int (\ln x)' \ln x \, dx$

$= (\ln x)^2 - \int \frac{\ln x}{x} dx$

$\therefore 2\int \frac{\ln x}{x} dx = (\ln x)^2 + C_1$

Considering the constant of integration,

$\int \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2 + C$

(3) $\int e^{2x} \sin 3x \, dx = \int \left(\frac{1}{2}e^{2x}\right)' \sin 3x \, dx$

$= \frac{1}{2}e^{2x}\sin 3x - \int \frac{1}{2}e^{2x}(\sin 3x)' dx$

$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x \, dx$

$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int \left(\frac{1}{2}e^{2x}\right)' \cos 3x \, dx$

$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\left[\frac{1}{2}e^{2x}\cos 3x - \int \frac{1}{2}e^{2x}(\cos 3x)' dx\right]$

$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}\int e^{2x}\sin 3x \, dx$

$\therefore \frac{13}{4}\int e^{2x}\sin 3x \, dx = \frac{1}{4}e^{2x}(2\sin 3x - 3\cos 3x) + C_1$

Considering the constant of integration,

$\int e^{2x}\sin 3x \, dx = \frac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x) + C$

087a

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Integration by Parts

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Find the following indefinite integrals using Integration by Substitution
Integration by Parts.

Ex. $\int e^{-\sqrt{x}}\,dx$

[Sol] Let $-\sqrt{x}=t$. Since $x=t^2$, $dx=2t\,dt$

$$\therefore \int e^{-\sqrt{x}}\,dx=\int e^t\cdot 2t\,dt$$
$$=2\int t(e^t)'\,dt$$
$$=2\left[te^t-\int (t)'e^t\,dt\right]$$
$$=2te^t-2\int e^t\,dt$$
$$=2te^t-2e^t+C$$
$$=2e^t(t-1)+C$$
$$=-2e^{-\sqrt{x}}(\sqrt{x}+1)+C$$

Substituting $-\sqrt{x}$ into t

(1) $\int e^{\sin x}\sin x\cos x\,dx$

[Sol] Let $\sin x=t$. $\cos x\,dx=dt$

$$\therefore \int e^{\sin x}\sin x\cos x\,dx=\int e^t\cdot t\,dt$$
$$=\int t(e^t)'\,dt$$
$$=te^t-\int (t)'e^t\,dt$$
$$=te^t-\int e^t\,dt$$
$$=te^t-e^t+C$$
$$=e^t(t-1)+C$$
$$=e^{\sin x}(\sin x-1)+C$$

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087b

2) $\int \frac{\ln x}{x^2} dx$

Sol] Let $\ln x = t$. $x = e^t$, $\frac{1}{x} dx = dt$

$$\begin{aligned}
\therefore \int \frac{\ln x}{x^2} dx &= \int \frac{t}{e^t} dt \\
&= \int t e^{-t} dt \\
&= \int t(-e^{-t})' dt \\
&= -te^{-t} - \int (t)'(-e^{-t}) dt \\
&= -te^{-t} + \int e^{-t} dt \\
&= -te^{-t} - e^{-t} + C \\
&= -\frac{1}{e^t}(t+1) + C \\
&= -\frac{1}{x}(\ln x + 1) + C
\end{aligned}$$

$e^t = x$

Alternative Solution

Let $\frac{1}{x} = t$.

$x = \frac{1}{t}$, $-\frac{1}{x^2} dx = dt$

Therefore,

$$\begin{aligned}
&\int \frac{\ln x}{x^2} dx \\
&= \int \ln \frac{1}{t} \cdot (-1) dt \\
&= -\int (t)' \ln \frac{1}{t} dt \\
&= -\left[t \ln \frac{1}{t} - \int t \left(\ln \frac{1}{t} \right)' dt \right] \\
&= -t \ln \frac{1}{t} - \int dt \\
&= -t \ln \frac{1}{t} - t + C \\
&= -t \left(\ln \frac{1}{t} + 1 \right) + C \\
&= -\frac{1}{x}(\ln x + 1) + C
\end{aligned}$$

3) $\int \frac{x}{1+x^2} \ln(1+x^2) dx$

Sol] Let $1 + x^2 = t$. $2x\,dx = dt$

$$\begin{aligned}
\therefore \int \frac{x}{1+x^2} \ln(1+x^2) dx &= \int \frac{1}{t} \ln t \cdot \frac{1}{2} dt \\
&= \frac{1}{2} \int \frac{\ln t}{t} dt \\
&= \frac{1}{2} \int \ln t (\ln t)' dt \\
&= \frac{1}{2} \left[(\ln t)^2 - \int (\ln t)' \ln t \, dt \right] \\
&= \frac{1}{2} (\ln t)^2 - \frac{1}{2} \int \frac{\ln t}{t} dt
\end{aligned}$$

$$\therefore \int \frac{\ln t}{t} dt = \frac{1}{2} (\ln t)^2 + C_1 = \frac{1}{2} [\ln(1+x^2)]^2 + C_1$$

Considering the constant of integration,

$$\int \frac{x}{1+x^2} \ln(1+x^2) dx = \frac{1}{2} \int \frac{\ln t}{t} dt = \frac{1}{4} [\ln(1+x^2)]^2 + C$$

This can also be solved by letting $\ln(1+x^2) = t$. However, since the form does not correspond to the given question, let $1 + x^2 = t$ instead.

Integration by Parts

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1. Given that n is a positive integer and $I_n=\int(\ln x)^n dx$, solve the following questions.

(1) Find $I_1=\int \ln x\,dx$.

[Sol] $I_1=\int(x)'\ln x\,dx$

$=x\ln x-\int x(\ln x)'dx$

$=x\ln x-\int dx$

$=x\ln x-x+C$

(2) Prove $I_n=x(\ln x)^n-nI_{n-1}$.

[Sol] $I_n=\int(x)'(\ln x)^n dx$

$=x(\ln x)^n-\int x[(\ln x)^n]'dx$

$=x(\ln x)^n-n\int(\ln x)^{n-1}dx$

$=x(\ln x)^n-nI_{n-1}$

$\int x[(\ln x)^n]'dx$

$=\int x\cdot n(\ln x)^{n-1}\cdot\frac{1}{x}dx$

$=n\int(\ln x)^{n-1}dx$

(3) Using the results from (1) and (2), find $I_3=\int(\ln x)^3 dx$.

[Sol] $I_3=x(\ln x)^3-3I_2$ ← From (2)

$=x(\ln x)^3-3[x(\ln x)^2-2I_1]$ ← From (2)

$=x(\ln x)^3-3x(\ln x)^2+6(x\ln x-x+C_1)$ ← From (1)

Considering the constant of integration,

$I_3=x(\ln x)^3-3x(\ln x)^2+6x\ln x-6x+C$

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O88b

2. Given that $I=\int e^x \sin x\,dx$ and $J=\int e^x \cos x\,dx$, solve the following questions.

(1) Prove $I=e^x \sin x - J$ and $J=e^x \cos x + I$.

[Sol] $I=\int (e^x)' \sin x\,dx$

$=e^x \sin x - \int e^x (\sin x)'\,dx$

$=e^x \sin x - \int e^x \cos x\,dx$

$=e^x \sin x - J$

$J=\int (e^x)' \cos x\,dx$

$=e^x \cos x - \int e^x (\cos x)'\,dx$

$=e^x \cos x + \int e^x \sin x\,dx$

$=e^x \cos x + I$

(2) Using the result from (1), find I and J.

[Sol] From (1),

$I=e^x \sin x - J \quad \cdots ①$

$J=e^x \cos x + I \quad \cdots ②$

Substituting ② into ①,

$I=e^x \sin x - (e^x \cos x + I)$

$2I=e^x(\sin x - \cos x)+C_1$

Considering the constant of integration,

$I=\frac{1}{2}e^x(\sin x - \cos x)+C$

Substituting ① into ②,

$J=e^x \cos x + (e^x \sin x - J)$

$2J=e^x(\cos x + \sin x)+C_2$

Considering the constant of integration,

$J=\frac{1}{2}e^x(\cos x + \sin x)+C$

Integration by Parts

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100%	~90%	~80%	~70%	69%~
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1. Given that n is an integer and $I_n=\int \sin^n x\,dx$, prove that the following equality is true.

   When $n \geqq 2$, $I_n=\frac{1}{n}[-\sin^{n-1}x\cos x+(n-1)I_{n-2}]$

[Sol] When $n \geqq 2$,

$$I_n=\int \sin^{n-1}x\cdot \sin x\,dx$$

$$=\int \sin^{n-1}x(-\cos x)'\,dx$$

$$=-\sin^{n-1}x\cos x-\int (\sin^{n-1}x)'(-\cos x)\,dx$$

$$=-\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}x\cos^2 x\,dx$$

$$=-\sin^{n-1}x\cos x+(n-1)\int \sin^{n-2}x(1-\sin^2 x)\,dx$$

$$=-\sin^{n-1}x\cos x+(n-1)\int (\sin^{n-2}x-\sin^n x)\,dx$$

$$=-\sin^{n-1}x\cos x+(n-1)(I_{n-2}-I_n)$$

$$\therefore\ nI_n=-\sin^{n-1}x\cos x+(n-1)I_{n-2}$$

$$\therefore\ I_n=\frac{1}{n}[-\sin^{n-1}x\cos x+(n-1)I_{n-2}]$$

$(\sin^{n-1}x)'(-\cos x)$
$=(n-1)\sin^{n-1-1}x\cdot(\sin x)'\cdot(-\cos x)$
$=-(n-1)\sin^{n-2}x\cos^2 x$

Transposing $-(n-1)I_n$ to the LHS

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O89b

2. Given that the function $f(x)$ satisfies $f'(x)=x\cos x$ and $f(0)=0$, solve the following questions.

(1) Find $f(x)$.

[Sol] Since $f'(x)=x\cos x$,

$$\begin{aligned} f(x) &= \int x\cos x\,dx \\ &= \int x(\sin x)'dx \\ &= x\sin x - \int (x)'\sin x\,dx \\ &= x\sin x - \int \sin x\,dx \\ &= x\sin x + \cos x + C \end{aligned}$$

Since $f(0)=0$, $C=-1$

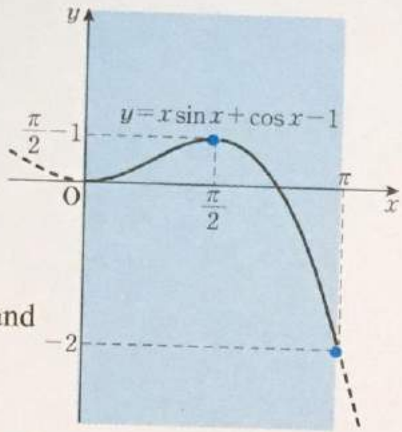
$\therefore$ $\boldsymbol{f(x)=x\sin x+\cos x-1}$

(2) Find the maximum and minimum values of $f(x)$ in $0 \leqslant x \leqslant \pi$.

[Sol] Since $f'(x)=x\cos x$,

when $f'(x)=0$ in $0<x<\pi$, $\cos x=0$, i.e. $x=\frac{\pi}{2}$

x	0	$\cdots$	$\frac{\pi}{2}$	$\cdots$	π
$f'(x)$		$+$	0	$-$	
$f(x)$	0	$\nearrow$	$\frac{\pi}{2}-1$	$\searrow$	-2



From the variation table,

the maximum value is $\frac{\pi}{2}-1$, at $x=\frac{\pi}{2}$ and

the minimum value is -2, at $x=\pi$.

O90a

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Integration by Parts

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Find the following indefinite integrals using Integration by Parts.

(1) ➡ O81

$$\int xe^{-x}dx=\int x(-e^{-x})'dx$$
$$=-xe^{-x}-\int(x)'(-e^{-x})dx$$
$$=-xe^{-x}+\int e^{-x}dx$$
$$=-xe^{-x}-e^{-x}+C$$
$$\left[=-(x+1)e^{-x}+C\right]$$

(2) ➡ O82

$$\int x^2\ln x\,dx=\int\left(\frac{1}{3}x^3\right)'\ln x\,dx$$
$$=\frac{1}{3}x^3\ln x-\int\frac{1}{3}x^3(\ln x)'dx$$
$$=\frac{1}{3}x^3\ln x-\frac{1}{3}\int x^2dx$$
$$=\frac{1}{3}x^3\ln x-\frac{1}{9}x^3+C$$
$$\left[=\frac{1}{9}x^3(3\ln x-1)+C\right]$$

(3) ➡ O83

$$\int\ln(x+3)dx=\int(x)'\ln(x+3)dx$$
$$=x\ln(x+3)-\int x[\ln(x+3)]'dx$$
$$=x\ln(x+3)-\int\frac{x}{x+3}dx$$
$$=x\ln(x+3)-\int\left(1-\frac{3}{x+3}\right)dx$$
$$=x\ln(x+3)-[x-3\ln(x+3)]+C$$
$$=(x+3)\ln(x+3)-x+C$$

[Alternative Solution

$$\int\ln(x+3)dx=\int(x+3)'\ln(x+3)dx=(x+3)\ln(x+3)-\int(x+3)[\ln(x+3)]'dx$$
$$=(x+3)\ln(x+3)-\int dx=(x+3)\ln(x+3)-x+C$$
]

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O90b

(4) $\int(x^2+1)\sin x\,dx=\int(x^2+1)(-\cos x)'dx$ ➡ O84

$$=-(x^2+1)\cos x-\int(x^2+1)'(-\cos x)dx$$
$$=-(x^2+1)\cos x+2\int x\cos x\,dx$$
$$=-(x^2+1)\cos x+2\int x(\sin x)'dx$$
$$=-(x^2+1)\cos x+2\left[x\sin x-\int(x)'\sin x\,dx\right]$$
$$=-(x^2+1)\cos x+2x\sin x-2\int\sin x\,dx$$
$$=-(x^2+1)\cos x+2x\sin x+2\cos x+C$$
$$=-(x^2-1)\cos x+2x\sin x+C$$

(5) $\int\cos(\ln x)dx=\int(x)'\cos(\ln x)dx$ ➡ O86

$$=x\cos(\ln x)-\int x[\cos(\ln x)]'dx$$
$$=x\cos(\ln x)+\int\sin(\ln x)dx$$
$$=x\cos(\ln x)+\int(x)'\sin(\ln x)dx$$
$$=x\cos(\ln x)+\left\{x\sin(\ln x)-\int x[\sin(\ln x)]'dx\right\}$$
$$=x\cos(\ln x)+x\sin(\ln x)-\int\cos(\ln x)dx$$

$$\therefore\ 2\int\cos(\ln x)dx=x[\cos(\ln x)+\sin(\ln x)]+C_1$$

Considering the constant of integration,

$$\int\cos(\ln x)dx=\frac{1}{2}x[\cos(\ln x)+\sin(\ln x)]+C$$

Definite Integrals

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If $F(x)$ is an indefinite integral (antiderivative) of $f(x)$, $F(b)-F(a)$ is called the ***definite integral*** of $f(x)$ from a to b and expressed as $\int_a^b f(x)dx$. Also, a and b are called the ***lower limit*** and the ***upper limit*** respectively.

Definite Integral

If $F(x)$ is an indefinite integral (antiderivative) of $f(x)$, then

$$\int_a^b f(x)dx=\Big[F(x)\Big]_a^b=F(b)-F(a)$$

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Find the following definite integrals.

Ex. $\int_3^4 \sqrt{x+1}\,dx=\left[\frac{2}{3}(x+1)^{\frac{3}{2}}\right]_3^4=\frac{2}{3}\left(5^{\frac{3}{2}}-4^{\frac{3}{2}}\right)=\frac{2}{3}(5\sqrt{5}-8)$

(1) $\int_1^2 \sqrt{x}\,dx=\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^2=\frac{2}{3}\left(2^{\frac{3}{2}}-1^{\frac{3}{2}}\right)=\frac{2}{3}(2\sqrt{2}-1)$

(2) $\int_0^2 \sqrt{2-x}\,dx=\left[-\frac{2}{3}(2-x)^{\frac{3}{2}}\right]_0^2=\frac{2}{3}\cdot 2^{\frac{3}{2}}=\frac{4\sqrt{2}}{3}$

(3) $\int_0^1 \sqrt{3t+1}\,dt=\left[\frac{2}{9}(3t+1)^{\frac{3}{2}}\right]_0^1=\frac{2}{9}\left(4^{\frac{3}{2}}-1^{\frac{3}{2}}\right)=\frac{14}{9}$

(4) $\int_{-1}^0 \sqrt{1-2u}\,du=\left[-\frac{1}{3}(1-2u)^{\frac{3}{2}}\right]_{-1}^0=-\frac{1}{3}\left(1^{\frac{3}{2}}-3^{\frac{3}{2}}\right)=\sqrt{3}-\frac{1}{3}$

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(5) $$\int_0^1 \frac{dx}{\sqrt{x}+\sqrt{x+1}}=\int_0^1 \frac{\sqrt{x}-\sqrt{x+1}}{(\sqrt{x}+\sqrt{x+1})(\sqrt{x}-\sqrt{x+1})}dx$$
$$=\int_0^1 (\sqrt{x+1}-\sqrt{x})dx$$
$$=\left[\frac{2}{3}(x+1)^{\frac{3}{2}}-\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$$
$$=\left(\frac{2}{3}\cdot 2^{\frac{3}{2}}-\frac{2}{3}\cdot 1^{\frac{3}{2}}\right)-\frac{2}{3}\cdot 1^{\frac{3}{2}}$$
$$=\boldsymbol{\frac{4}{3}(\sqrt{2}-1)}$$

(6) $$\int_0^4 \frac{dx}{\sqrt{x+5}-\sqrt{x}}=\int_0^4 \frac{\sqrt{x+5}+\sqrt{x}}{(\sqrt{x+5}-\sqrt{x})(\sqrt{x+5}+\sqrt{x})}dx$$
$$=\frac{1}{5}\int_0^4 (\sqrt{x+5}+\sqrt{x})dx$$
$$=\frac{1}{5}\left[\frac{2}{3}(x+5)^{\frac{3}{2}}+\frac{2}{3}x^{\frac{3}{2}}\right]_0^4$$
$$=\frac{1}{5}\left[\left(\frac{2}{3}\cdot 9^{\frac{3}{2}}+\frac{2}{3}\cdot 4^{\frac{3}{2}}\right)-\frac{2}{3}\cdot 5^{\frac{3}{2}}\right]$$
$$=\boldsymbol{\frac{2}{3}(7-\sqrt{5})}$$

(7) $$\int_0^1 \frac{2x+1}{\sqrt{3x+1}-\sqrt{x}}dx=\int_0^1 \frac{(2x+1)(\sqrt{3x+1}+\sqrt{x})}{(\sqrt{3x+1}-\sqrt{x})(\sqrt{3x+1}+\sqrt{x})}dx$$
$$=\int_0^1 \frac{(2x+1)(\sqrt{3x+1}+\sqrt{x})}{2x+1}dx$$
$$=\int_0^1 (\sqrt{3x+1}+\sqrt{x})dx$$
$$=\left[\frac{2}{9}(3x+1)^{\frac{3}{2}}+\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$$
$$=\left(\frac{2}{9}\cdot 4^{\frac{3}{2}}+\frac{2}{3}\cdot 1^{\frac{3}{2}}\right)-\frac{2}{9}\cdot 1^{\frac{3}{2}}$$
$$=\boldsymbol{\frac{20}{9}}$$

Definite Integrals

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Find the following definite integrals.

Ex.

$$\int_1^2 \frac{2x^2+\sqrt{x}-1}{x}dx=\int_1^2\left(2x+x^{-\frac{1}{2}}-\frac{1}{x}\right)dx$$
$$=\left[x^2+2x^{\frac{1}{2}}-\ln|x|\right]_1^2$$
$$=(4+2\sqrt{2}-\ln 2)-(1+2-\ln 1)$$
$$=1+2\sqrt{2}-\ln 2$$

$\ln 1=0$

(1) $$\int_1^2 \frac{x^2+1}{x^3}dx=\int_1^2\left(\frac{1}{x}+\frac{1}{x^3}\right)dx$$
$$=\left[\ln|x|-\frac{1}{2x^2}\right]_1^2$$
$$=\left(\ln 2-\frac{1}{8}\right)-\left(\ln 1-\frac{1}{2}\right)$$
$$=\ln 2+\frac{3}{8}$$

(2) $$\int_1^4 \frac{t+1}{\sqrt{t}}dt=\int_1^4\left(t^{\frac{1}{2}}+t^{-\frac{1}{2}}\right)dt$$
$$=\left[\frac{2}{3}t^{\frac{3}{2}}+2t^{\frac{1}{2}}\right]_1^4$$
$$=\left(\frac{16}{3}+4\right)-\left(\frac{2}{3}+2\right)$$
$$=\frac{20}{3}$$

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(3) $\displaystyle\int_{-1}^{0}\frac{x^2+2x+1}{x+2}\,dx=\int_{-1}^{0}\left(\boxed{x}+\frac{1}{x+2}\right)dx$ ⟵

$$\begin{array}{r} x \\ x+2\,\overline{)\,x^2+2x+1} \\ \underline{x^2+2x} \\ 1 \end{array}$$

$$=\left[\frac{1}{2}x^2+\ln|x+2|\right]_{-1}^{0}$$

$$=\ln 2-\left(\frac{1}{2}+\ln 1\right)$$

$$=\mathbf{\ln 2-\frac{1}{2}}$$

(4) $\displaystyle\int_{0}^{1}\frac{x^2+3x+1}{x+1}\,dx=\int_{0}^{1}\left(x+2-\frac{1}{x+1}\right)dx$

$$=\left[\frac{1}{2}x^2+2x-\ln|x+1|\right]_{0}^{1}$$

$$=\left(\frac{1}{2}+2-\ln 2\right)-(-\ln 1)$$

$$=\mathbf{\frac{5}{2}-\ln 2}$$

(5) $\displaystyle\int_{0}^{1}\frac{(x+3)^2}{x+1}\,dx=\int_{0}^{1}\frac{x^2+6x+9}{x+1}\,dx$

$$=\int_{0}^{1}\left(x+5+\frac{4}{x+1}\right)dx$$

$$=\left[\frac{1}{2}x^2+5x+4\ln|x+1|\right]_{0}^{1}$$

$$=\left(\frac{1}{2}+5+4\ln 2\right)-4\ln 1$$

$$=\mathbf{\frac{11}{2}+4\ln 2}$$

Definite Integrals

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Find the following definite integrals.

Ex.

$$\int_0^1 \frac{dx}{(x+1)(x+2)}$$

[Sol] Let $\frac{1}{(x+1)(x+2)}=\frac{a}{x+1}+\frac{b}{x+2}$.

Rearranging into an integrable form

Multiplying both sides by $(x+1)(x+2)$

$$1=a(x+2)+b(x+1)$$
$$=(a+b)x+(2a+b)$$
$$\therefore \begin{cases} a+b=0 \\ 2a+b=1 \end{cases}$$
$$\therefore a=1,\ b=-1$$

Therefore,

$$\int_0^1 \frac{dx}{(x+1)(x+2)}=\int_0^1\left(\frac{1}{x+1}-\frac{1}{x+2}\right)dx$$
$$=\Big[\ln|x+1|-\ln|x+2|\Big]_0^1$$
$$=\left[\ln\left|\frac{x+1}{x+2}\right|\right]_0^1=\ln\frac{2}{3}-\ln\frac{1}{2}=\ln\frac{4}{3}$$

$\ln M-\ln N=\ln\frac{M}{N}$

(1) $\displaystyle\int_1^3 \frac{dx}{x(x+1)}$

[Sol] Let $\frac{1}{x(x+1)}=\frac{a}{x}+\frac{b}{x+1}$.

$$1=a(x+1)+bx$$
$$=(a+b)x+a$$
$$\therefore \begin{cases} a+b=0 \\ a=1 \end{cases}$$
$$\therefore a=1,\ b=-1$$

Therefore,

$$\int_1^3 \frac{dx}{x(x+1)}=\int_1^3\left(\frac{1}{x}-\frac{1}{x+1}\right)dx$$
$$=\Big[\ln|x|-\ln|x+1|\Big]_1^3$$
$$=\left[\ln\left|\frac{x}{x+1}\right|\right]_1^3=\ln\frac{3}{4}-\ln\frac{1}{2}=\ln\frac{3}{2}$$

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(2) $\displaystyle\int_0^1 \frac{2x-7}{(x+1)(x-2)}dx$

[Sol] Let $\dfrac{2x-7}{(x+1)(x-2)}=\dfrac{a}{x+1}+\dfrac{b}{x-2}$.

$$2x-7=a(x-2)+b(x+1)$$
$$=(a+b)x-(2a-b)$$
$$\therefore \begin{cases} a+b=2 \\ 2a-b=7 \end{cases}$$
$$\therefore\ a=3,\ b=-1$$

Therefore,

$$\int_0^1 \frac{2x-7}{(x+1)(x-2)}dx=\int_0^1\left(\frac{3}{x+1}-\frac{1}{x-2}\right)dx$$
$$=\Big[3\ln|x+1|-\ln|x-2|\Big]_0^1$$
$$=\left[\ln\left|\frac{(x+1)^3}{x-2}\right|\right]_0^1=\ln 8-\ln\frac{1}{2}=4\ln 2$$

$n\ln M=\ln M^n$

(3) $\displaystyle\int_{-3}^1 \frac{x}{x^2+2x-8}dx$

[Sol] Let $\dfrac{x}{x^2+2x-8}=\dfrac{x}{(x+4)(x-2)}=\dfrac{a}{x+4}+\dfrac{b}{x-2}$.

$$x=a(x-2)+b(x+4)$$
$$=(a+b)x-(2a-4b)$$
$$\therefore \begin{cases} a+b=1 \\ 2a-4b=0 \end{cases}$$
$$\therefore\ a=\frac{2}{3},\ b=\frac{1}{3}$$

Therefore,

$$\int_{-3}^1 \frac{x}{x^2+2x-8}dx=\frac{1}{3}\int_{-3}^1\left(\frac{2}{x+4}+\frac{1}{x-2}\right)dx$$
$$=\frac{1}{3}\Big[2\ln|x+4|+\ln|x-2|\Big]_{-3}^1$$
$$=\frac{1}{3}\Big[\ln|(x+4)^2(x-2)|\Big]_{-3}^1$$
$$=\frac{1}{3}(\ln 25-\ln 5)=\mathbf{\frac{1}{3}\ln 5}$$

$\ln M+\ln N$
$=\ln MN$

Definite Integrals

Name

Date / /

Time : to :

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Find the following definite integrals.

Ex.

$$\int_0^1 e^{2x}dx=\left[\frac{1}{2}e^{2x}\right]_0^1$$
$$=\frac{1}{2}(e^2-e^0)$$
$$=\frac{1}{2}(e^2-1)$$

(1) $$\int_{-1}^1 e^{3x}dx=\left[\frac{1}{3}e^{3x}\right]_{-1}^1$$
$$=\frac{1}{3}(e^3-e^{-3})$$
$$=\frac{1}{3}\left(e^3-\frac{1}{e^3}\right)$$

(2) $$\int_1^2\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)dx=\left[2e^{\frac{x}{2}}+2e^{-\frac{x}{2}}\right]_1^2$$
$$=(2e^1+2e^{-1})-\left(2e^{\frac{1}{2}}+2e^{-\frac{1}{2}}\right)$$
$$=2\left(e+\frac{1}{e}-\sqrt{e}-\frac{\sqrt{e}}{e}\right)\quad\left[=2\left(e+\frac{1}{e}-\sqrt{e}-\frac{1}{\sqrt{e}}\right)\right]$$

(3) $$\int_0^1(e^x-e^{-x})^2dx=\int_0^1(e^{2x}-2+e^{-2x})dx$$
$$=\left[\frac{1}{2}e^{2x}-2x-\frac{1}{2}e^{-2x}\right]_0^1$$
$$=\left(\frac{1}{2}e^2-2-\frac{1}{2}e^{-2}\right)-\left(\frac{1}{2}e^0-\frac{1}{2}e^0\right)$$
$$=\frac{1}{2}e^2-\frac{1}{2e^2}-2\quad\left[=\frac{1}{2}\left(e^2-\frac{1}{e^2}-4\right)\right]$$

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Ex.

$$\int_0^1 3^x dx=\left[\frac{3^x}{\ln 3}\right]_0^1$$
$$=\frac{3^1-3^0}{\ln 3}$$
$$=\frac{2}{\ln 3}$$

(4) $$\int_{-3}^0 2^x dx=\left[\frac{2^x}{\ln 2}\right]_{-3}^0$$
$$=\frac{2^0-2^{-3}}{\ln 2}$$
$$=\frac{7}{8\ln 2}$$

(5) $$\int_0^2 2^{3t} dt=\left[\frac{2^{3t}}{3\ln 2}\right]_0^2$$
$$=\frac{2^6-2^0}{3\ln 2}$$
$$=\frac{21}{\ln 2}$$

(6) $$\int_1^2 5^{-x} dx=\left[-\frac{5^{-x}}{\ln 5}\right]_1^2$$
$$=-\frac{5^{-2}-5^{-1}}{\ln 5}$$
$$=\frac{4}{25\ln 5}$$

Definite Integrals

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Time : to :

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Find the following definite integrals.

Ex.

$$\int_0^{\pi} \sin x\,dx = \Big[-\cos x\Big]_0^{\pi}$$
$$= -(\cos\pi - \cos 0)$$
$$= 2$$

(1) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x\,dx = \Big[\sin x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$$= \sin\frac{\pi}{2} - \sin\frac{\pi}{6}$$
$$= \frac{1}{2}$$

(2) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\cos^2 x} = \Big[\tan x\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$$= \tan\frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right)$$
$$= 2$$

(3) $\displaystyle\int_0^{\frac{\pi}{2}} \sin 2\theta\,d\theta = \Big[-\frac{1}{2}\cos 2\theta\Big]_0^{\frac{\pi}{2}}$

$$= -\frac{1}{2}(\cos\pi - \cos 0)$$
$$= 1$$

(4) $\displaystyle\int_0^{\frac{\pi}{2}} (2\sin x + \cos 2x)\,dx = \Big[-2\cos x + \frac{1}{2}\sin 2x\Big]_0^{\frac{\pi}{2}}$

$$= \left(-2\cos\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) - \left(-2\cos 0 + \frac{1}{2}\sin 0\right)$$
$$= 2$$

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(5) $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin^2 x}{1+\cos x}\,dx = \int_0^{\frac{\pi}{3}} \frac{1-\cos^2 x}{1+\cos x}\,dx$

$$= \int_0^{\frac{\pi}{3}} \frac{(1+\cos x)(1-\cos x)}{1+\cos x}\,dx$$

$$= \int_0^{\frac{\pi}{3}} (1-\cos x)\,dx$$

$$= \Big[\, x - \sin x \,\Big]_0^{\frac{\pi}{3}}$$

$$= \left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) - (-\sin 0)$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

(6) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos^2 x}\,dx$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\cos^2 x}{\cos^2 x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x} - 1\right) dx$$

$$= \Big[\, \tan x - x \,\Big]_0^{\frac{\pi}{4}}$$

$$= \left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - \tan 0$$

$$= 1 - \frac{\pi}{4}$$

(7) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\boxed{1+\sin x}}{1+\sin x}\,dx$

$$= \int_0^{\frac{\pi}{2}} dx$$

$$= \Big[\, x \,\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

Definite Integrals

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Find the following definite integrals.

Ex.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1-\cos 2x)\,dx \quad \Leftarrow \quad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{1}{2}\right)\right]$$

$$= \frac{\pi}{8} + \frac{1}{4}$$

(1) $$\int_{-\pi}^{\pi} \cos^2 x\,dx = \frac{1}{2}\int_{-\pi}^{\pi} (1+\cos 2x)\,dx \quad \Leftarrow \quad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}[\pi - (-\pi)]$$

$$= \pi$$

(2) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin\theta + \cos\theta)^2\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta)\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (1 + \sin 2\theta)\,d\theta \quad \Leftarrow \quad 2\sin\alpha\cos\alpha = \sin 2\alpha$$

$$= \left[\theta - \frac{1}{2}\cos 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} - \left(\frac{\pi}{6} - \frac{1}{4}\right)$$

$$= \frac{\pi}{12} + \frac{1}{4}$$

【Reference】064

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Ex.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 3x \cos x \, dx$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sin 4x + \sin 2x)\, dx$$

$$= \frac{1}{2}\left[-\frac{1}{4}\cos 4x - \frac{1}{2}\cos 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[\left(\frac{1}{8}+\frac{1}{4}\right)-\left(\frac{1}{8}-\frac{1}{4}\right)\right]$$

$$= \frac{1}{4}$$

(3) $\displaystyle\int_0^{\pi} \cos 2x \sin x \, dx$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$

$$= \frac{1}{2}\int_0^{\pi} (\sin 3x - \sin x)\, dx$$

$$= \frac{1}{2}\left[-\frac{1}{3}\cos 3x + \cos x\right]_0^{\pi}$$

$$= \frac{1}{2}\left[\left(\frac{1}{3}-1\right)-\left(-\frac{1}{3}+1\right)\right]$$

$$= -\frac{2}{3}$$

(4) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 3x \cos 2x \, dx$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos 5x + \cos x)\, dx$$

$$= \frac{1}{2}\left[\frac{1}{5}\sin 5x + \sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\frac{1}{5}+1\right)-\left(-\frac{1}{5}-1\right)\right]$$

$$= \frac{6}{5}$$

【Reference】066

Definite Integrals

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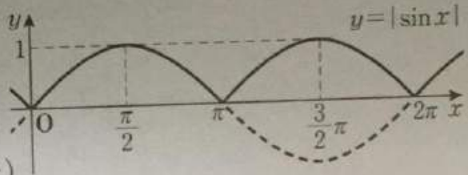
Definite integrals have the following property.

Property of Definite Integrals I

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$$

Find the following definite integrals.

Ex. $\displaystyle\int_0^{2\pi}|\sin x|dx$



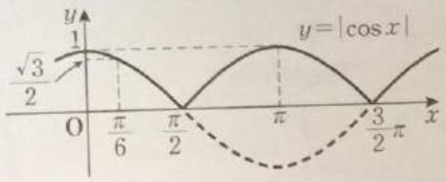
[Sol] $|\sin x|=\begin{cases}\sin x & (0\leqq x\leqq\pi)\\ -\sin x & (\pi\leqq x\leqq 2\pi)\end{cases}$

Therefore,

$$\int_0^{2\pi}|\sin x|dx=\int_0^{\pi}\sin x\,dx+\int_{\pi}^{2\pi}(-\sin x)dx$$
$$=\Big[-\cos x\Big]_0^{\pi}+\Big[\cos x\Big]_{\pi}^{2\pi}$$
$$=[1-(-1)]+[1-(-1)]$$
$$=4$$

$\displaystyle\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$

(1) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{3}{2}\pi}|\cos x|dx$



[Sol] $|\cos x|=\begin{cases}\cos x & \left(\frac{\pi}{6}\leqq x\leqq\frac{\pi}{2}\right)\\ -\cos x & \left(\frac{\pi}{2}\leqq x\leqq\frac{3}{2}\pi\right)\end{cases}$

Therefore,

$$\int_{\frac{\pi}{6}}^{\frac{3}{2}\pi}|\cos x|dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos x\,dx+\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}(-\cos x)dx$$
$$=\Big[\sin x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}+\Big[-\sin x\Big]_{\frac{\pi}{2}}^{\frac{3}{2}\pi}$$
$$=\left(1-\frac{1}{2}\right)+[1-(-1)]$$
$$=\frac{5}{2}$$

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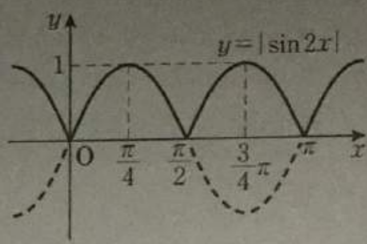
(2) $\displaystyle\int_0^{\pi} |\sin 2x|\,dx$

[Sol] $|\sin 2x| = \begin{cases} \sin 2x & \left(0 \leqslant x \leqslant \dfrac{\pi}{2}\right) \\ -\sin 2x & \left(\dfrac{\pi}{2} \leqslant x \leqslant \pi\right) \end{cases}$



Therefore,

$$\int_0^{\pi} |\sin 2x|\,dx = \int_0^{\frac{\pi}{2}} \sin 2x\,dx + \int_{\frac{\pi}{2}}^{\pi} (-\sin 2x)\,dx$$

$$= \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}} + \left[\frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\frac{1}{2}(-1-1) + \frac{1}{2}[1-(-1)]$$

$$= 2$$

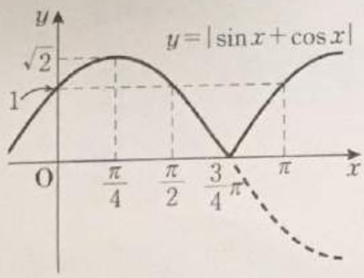
(3) $\displaystyle\int_0^{\pi} |\sin x + \cos x|\,dx$

[Sol] $\displaystyle\int_0^{\pi} |\sin x + \cos x|\,dx = \int_0^{\pi} \left|\sqrt{2}\sin\left(x + \boxed{\frac{\pi}{4}}\right)\right| dx \Leftarrow$ $a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\sin(\theta+\alpha)$

$$\left|\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right| = \begin{cases} \sqrt{2}\sin\left(x+\dfrac{\pi}{4}\right) & \left(0 \leqslant x \leqslant \dfrac{3}{4}\pi\right) \\ -\sqrt{2}\sin\left(x+\dfrac{\pi}{4}\right) & \left(\dfrac{3}{4}\pi \leqslant x \leqslant \pi\right) \end{cases}$$

Therefore,

$$\int_0^{\pi} |\sin x + \cos x|\,dx = \int_0^{\frac{3}{4}\pi} \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)dx + \int_{\frac{3}{4}\pi}^{\pi} \left[-\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right]dx$$



$$= \sqrt{2}\left[-\cos\left(x+\frac{\pi}{4}\right)\right]_0^{\frac{3}{4}\pi} + \sqrt{2}\left[\cos\left(x+\frac{\pi}{4}\right)\right]_{\frac{3}{4}\pi}^{\pi}$$

$$= \sqrt{2}\left[1-\left(-\frac{\sqrt{2}}{2}\right)\right] + \sqrt{2}\left[-\frac{\sqrt{2}}{2}-(-1)\right]$$

$$= 2\sqrt{2}$$

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Definite Integrals

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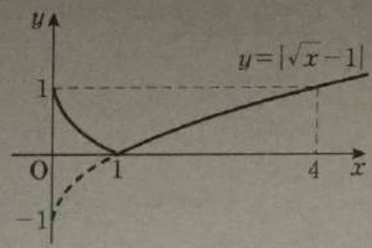
Find the following definite integrals.

(1) $\int_0^4 |\sqrt{x}-1|\,dx$



[Sol] $|\sqrt{x}-1|=\begin{cases} -(\sqrt{x}-1) & (0\leqslant x\leqslant 1) \\ \sqrt{x}-1 & (1\leqslant x\leqslant 4) \end{cases}$

Therefore,

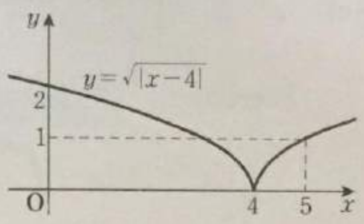
$$\int_0^4 |\sqrt{x}-1|\,dx=\int_0^1 [-(\sqrt{x}-1)]\,dx+\int_1^4 (\sqrt{x}-1)\,dx$$

$$=\left[-\frac{2}{3}x^{\frac{3}{2}}+x\right]_0^1+\left[\frac{2}{3}x^{\frac{3}{2}}-x\right]_1^4$$

$$=\left(-\frac{2}{3}+1\right)+\left[\left(\frac{16}{3}-4\right)-\left(\frac{2}{3}-1\right)\right]$$

$$=\mathbf{2}$$

(2) $\int_0^5 \sqrt{|x-4|}\,dx$



[Sol] $\sqrt{|x-4|}=\begin{cases} \sqrt{-(x-4)} & (0\leqslant x\leqslant 4) \\ \sqrt{x-4} & (4\leqslant x\leqslant 5) \end{cases}$

Therefore,

$$\int_0^5 \sqrt{|x-4|}\,dx=\int_0^4 \sqrt{-(x-4)}\,dx+\int_4^5 \sqrt{x-4}\,dx$$

$$=\left[-\frac{2}{3}(-x+4)^{\frac{3}{2}}\right]_0^4+\left[\frac{2}{3}(x-4)^{\frac{3}{2}}\right]_4^5$$

$$=\frac{16}{3}+\frac{2}{3}$$

$$=\mathbf{6}$$

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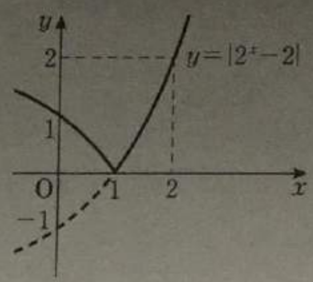
(3) $\int_0^2 |2^x-2|\,dx$

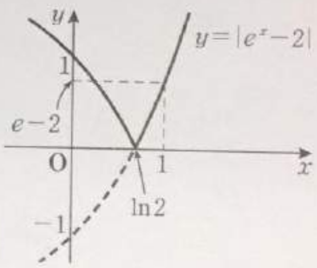


[Sol] $|2^x-2|=\begin{cases}-(2^x-2) & (0\leq x\leq 1)\\ 2^x-2 & (1\leq x\leq 2)\end{cases}$

Therefore,

$$\begin{aligned}\int_0^2|2^x-2|\,dx&=\int_0^1[-(2^x-2)]\,dx+\int_1^2(2^x-2)\,dx\\&=\left[-\frac{2^x}{\ln 2}+2x\right]_0^1+\left[\frac{2^x}{\ln 2}-2x\right]_1^2\\&=\left[\left(-\frac{2}{\ln 2}+2\right)-\left(-\frac{1}{\ln 2}\right)\right]+\left[\left(\frac{4}{\ln 2}-4\right)-\left(\frac{2}{\ln 2}-2\right)\right]\\&=\frac{1}{\ln 2}\end{aligned}$$

(4) $\int_0^1 |e^x-2|\,dx$



[Sol] $|e^x-2|=\begin{cases}-(e^x-2) & (0\leq x\leq \ln 2)\\ e^x-2 & (\ln 2\leq x\leq 1)\end{cases}$

Therefore, When $e^x-2=0$, $e^x=2$; therefore, $x=\ln 2$

$$\begin{aligned}\int_0^1|e^x-2|\,dx&=\int_0^{\ln 2}[-(e^x-2)]\,dx+\int_{\ln 2}^1(e^x-2)\,dx\\&=\left[-e^x+2x\right]_0^{\ln 2}+\left[e^x-2x\right]_{\ln 2}^1\\&=[(-e^{\ln 2}+2\ln 2)-(-1)]+[(e-2)-(e^{\ln 2}-2\ln 2)]\\&=4\ln 2+e-5\end{aligned}$$

When $e^{\ln 2}=a$, $\ln 2=\ln a$; therefore, $a=2$

Definite Integrals

Name

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1. Given that a and b are real numbers, find the minimum value of the definite integral $J=\int_0^{\pi}(a\sin x+b\sin 3x-1)^2dx$ and the corresponding values of a and b.

[Sol] $(a\sin x+b\sin 3x-1)^2$

$$=a^2\sin^2 x+b^2\sin^2 3x+1+2ab\sin x\sin 3x-2b\sin 3x-2a\sin x$$

$$=\frac{a^2}{2}(1-\cos 2x)+\frac{b^2}{2}(1-\cos 6x)+1$$

$$-ab[\cos 4x-\cos(-2x)]-2b\sin 3x-2a\sin x$$

※1

$$=\frac{a^2}{2}+\frac{b^2}{2}+1-\frac{a^2}{2}\cos 2x+ab\cos 2x-ab\cos 4x$$

$$-\frac{b^2}{2}\cos 6x-2a\sin x-2b\sin 3x$$

※2

$$\therefore\ J=\left[\left(\frac{a^2}{2}+\frac{b^2}{2}+1\right)x-\frac{a^2}{4}\sin 2x+\frac{ab}{2}\sin 2x-\frac{ab}{4}\sin 4x\right.$$

$$\left.-\frac{b^2}{12}\sin 6x+2a\cos x+\frac{2}{3}b\cos 3x\right]_0^{\pi}$$

$$=\left[\left(\frac{a^2}{2}+\frac{b^2}{2}+1\right)\pi-2a-\frac{2}{3}b\right]-\left(2a+\frac{2}{3}b\right)$$

$$=\frac{1}{2}\pi a^2-4a+\frac{1}{2}\pi b^2-\frac{4}{3}b+\pi$$

$$=\frac{\pi}{2}\left(a-\frac{4}{\pi}\right)^2+\frac{\pi}{2}\left(b-\frac{4}{3\pi}\right)^2+\pi-\frac{80}{9\pi}$$

※3

Therefore,

the minimum value of J is $\pi-\frac{80}{9\pi}$, **at** $a=\frac{4}{\pi}$ **and** $b=\frac{4}{3\pi}$.

※1 $\sin^2\alpha=\frac{1-\cos 2\alpha}{2}$, $\sin\alpha\sin\beta=-\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)]$

※2 $\cos(-\theta)=\cos\theta$

※3 The value is a minimum at $a-\frac{4}{\pi}=0$ and $b-\frac{4}{3\pi}=0$.

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2. Find $a_n=\int_0^{2\pi}|\sin t|\cos nt\,dt$. ($n$ is a natural number)

[Sol] $|\sin t|=\begin{cases}\sin t & (0\leqslant t\leqslant\pi)\\ -\sin t & (\pi\leqslant t\leqslant 2\pi)\end{cases}$

$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$

Therefore,

$$a_n=\int_0^{\pi}\sin t\cos nt\,dt+\int_{\pi}^{2\pi}(-\sin t)\cos nt\,dt$$
$$=\frac{1}{2}\int_0^{\pi}[\sin(n+1)t+\sin(1-n)t]dt-\frac{1}{2}\int_{\pi}^{2\pi}[\sin(n+1)t+\sin(1-n)t]dt$$

(i) When $n=1$,

The value of $\int\sin(1-n)t\,dt$ is different only when $n=1$.

$$a_1=\frac{1}{2}\int_0^{\pi}\sin 2t\,dt-\frac{1}{2}\int_{\pi}^{2\pi}\sin 2t\,dt$$
$$=\frac{1}{2}\left[-\frac{1}{2}\cos 2t\right]_0^{\pi}-\frac{1}{2}\left[-\frac{1}{2}\cos 2t\right]_{\pi}^{2\pi}$$
$$=-\frac{1}{4}(1-1)+\frac{1}{4}(1-1)=0$$

(ii) When $n\geqslant 2$,

$$a_n=\frac{1}{2}\left[-\frac{1}{n+1}\cos(n+1)t-\frac{1}{1-n}\cos(1-n)t\right]_0^{\pi}$$
$$-\frac{1}{2}\left[-\frac{1}{n+1}\cos(n+1)t-\frac{1}{1-n}\cos(1-n)t\right]_{\pi}^{2\pi}$$

$\cos 0=1$

$$=\frac{1}{2}\left\{\left[-\frac{1}{n+1}\cos(n+1)\pi-\frac{1}{1-n}\cos(1-n)\pi\right]-\left(-\frac{1}{n+1}-\frac{1}{1-n}\right)\right\}$$
$$-\frac{1}{2}\left\{\left(-\frac{1}{n+1}-\frac{1}{1-n}\right)-\left[-\frac{1}{n+1}\cos(n+1)\pi-\frac{1}{1-n}\cos(1-n)\pi\right]\right\}$$
$$=\frac{1-\cos(n+1)\pi}{n+1}+\frac{1-\cos(1-n)\pi}{1-n}$$

When k is an integer, $\cos 2k\pi=1$

When n is an even number, $\cos(n+1)\pi=\cos(1-n)\pi=-1$; therefore,

$$a_n=\frac{1-(-1)}{n+1}+\frac{1-(-1)}{1-n}=-\frac{4}{(n+1)(n-1)}$$

When n is an even number, $n+1$ and $1-n$ are odd numbers.

When n is an odd number, $\cos(n+1)\pi=\cos(1-n)\pi=1$; therefore,

$$a_n=\frac{1-1}{n+1}+\frac{1-1}{1-n}=0$$

When n is an odd number, $n+1$ and $1-n$ are even numbers.

From (i) and (ii), **when n is an even number, $a_n=-\frac{4}{(n+1)(n-1)}$** and **when n is an odd number, $a_n=0$.**

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Definite Integrals

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Find the following definite integrals.

(1) $\displaystyle\int_1^2 \sqrt{2x-1}\,dx=\left[\frac{1}{3}(2x-1)^{\frac{3}{2}}\right]_1^2$ ➡ O91

$\displaystyle=\frac{1}{3}(3\sqrt{3}-1)$

$\displaystyle=\sqrt{3}-\frac{1}{3}$

(2) $\displaystyle\int_1^e \frac{x^2-2}{x^3}\,dx=\int_1^e\left(\frac{1}{x}-\frac{2}{x^3}\right)dx$ ➡ O92

$\displaystyle=\left[\ln|x|+\frac{1}{x^2}\right]_1^e$

$\displaystyle=\left(\ln e+\frac{1}{e^2}\right)-(\ln 1+1)$

$\displaystyle=\frac{1}{e^2}$

(3) $\displaystyle\int_0^1 (e^x-e^{-x})\,dx=\left[e^x+e^{-x}\right]_0^1$ ➡ O94

$=(e^1+e^{-1})-(e^0+e^0)$

$\displaystyle=e+\frac{1}{e}-2$

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(4) $\displaystyle\int_0^1 7^x dx=\left[\frac{7^x}{\ln 7}\right]_0^1$ ➡ 094

$\displaystyle=\frac{7^1-7^0}{\ln 7}$

$\displaystyle=\mathbf{\frac{6}{\ln 7}}$

(5) $\displaystyle\int_0^{\frac{\pi}{3}}(\cos 2x-\sin^2 x)dx=\int_0^{\frac{\pi}{3}}\left[\cos 2x-\frac{1}{2}(1-\cos 2x)\right]dx$ ➡ 096

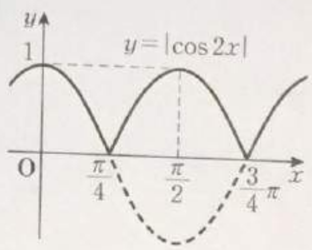
$\displaystyle=\int_0^{\frac{\pi}{3}}\left(\frac{3}{2}\cos 2x-\frac{1}{2}\right)dx$

$\displaystyle=\left[\frac{3}{4}\sin 2x-\frac{1}{2}x\right]_0^{\frac{\pi}{3}}$

$\displaystyle=\mathbf{\frac{3\sqrt{3}}{8}-\frac{\pi}{6}}$

(6) $\displaystyle\int_0^{\frac{\pi}{2}}|\cos 2x|dx$ ➡ 097

[Sol] $|\cos 2x|=\begin{cases}\cos 2x & \left(0\leqslant x\leqslant\frac{\pi}{4}\right)\\ -\cos 2x & \left(\frac{\pi}{4}\leqslant x\leqslant\frac{\pi}{2}\right)\end{cases}$



Therefore,

$\displaystyle\int_0^{\frac{\pi}{2}}|\cos 2x|dx=\int_0^{\frac{\pi}{4}}\cos 2x\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(-\cos 2x)dx$

$\displaystyle=\left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}+\left[-\frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$\displaystyle=\frac{1}{2}+\frac{1}{2}$

$=\mathbf{1}$

Integration by Substitution for Definite Integrals

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If the function $f(x)$ is continuous on the interval $[a, b]$ and x can be expressed as $x=g(t)$ where the function $g(t)$ is differentiable, then from Integration by Substitution, the indefinite integral of $f(x)$ is

$$\int f(x)dx=\int f(g(t))g'(t)dt \quad \cdots①$$

x	$a \longrightarrow b$
t	$\alpha \longrightarrow \beta$

If $x=g(t)$ changes from a to b, assume that t changes from α to β. This relationship is as shown on the right.

Then, let an indefinite integral of the function $f(x)$ be $F(x)$. From ①,

$$\begin{aligned}\int_\alpha^\beta f(g(t))g'(t)dt&=\Big[F(g(t))\Big]_\alpha^\beta\\&=F(g(\beta))-F(\boxed{g(\alpha)})\\&=F(b)-\boxed{F(a)}\\&=\Big[F(x)\Big]_{\boxed{a}}^b=\int_{\boxed{a}}^b f(x)dx\end{aligned}$$

Answers: $g(\alpha)$, $F(a)$, a, a

Integration by Substitution for Definite Integrals

When $x=g(t)$, if $a=g(\alpha)$ and $b=g(\beta)$, then

$$\int_a^b f(x)dx=\int_\alpha^\beta f(g(t))g'(t)dt$$

x	$a \longrightarrow b$
t	$\alpha \longrightarrow \beta$

This method is called ***Integration by Substitution for Definite Integrals.***

Find the following definite integrals using Integration by Substitution.

Ex. $\displaystyle\int_0^2 (2x-1)^3dx$

The interval of integration changes.

x	$0 \longrightarrow 2$
t	$-1 \longrightarrow 3$

[Sol] Let $2x-1=t$. Since $x=\dfrac{t+1}{2}$, $dx=\dfrac{1}{2}dt$

$$\therefore \int_0^2(2x-1)^3dx=\int_{\boxed{-1}}^3 t^3\cdot\frac{1}{2}dt=\frac{1}{2}\int_{\boxed{-1}}^3 t^3dt=\frac{1}{2}\left[\frac{1}{4}t^4\right]_{\boxed{-1}}^3=\boxed{10}$$

Answers: -1, -1, -1, 10

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(1) $\int_{-1}^{0}(3x+2)^4dx$

[Sol] Let $3x+2=t$. Since $x=\frac{t-2}{3}$, $dx=\frac{1}{3}dt$

x	$-1 \longrightarrow 0$
t	$-1 \longrightarrow 2$

$$\therefore \int_{-1}^{0}(3x+2)^4dx=\int_{-1}^{2}t^4\cdot\frac{1}{3}dt=\frac{1}{3}\int_{-1}^{2}t^4dt=\frac{1}{3}\left[\frac{1}{5}t^5\right]_{-1}^{2}=\frac{11}{5}$$

(2) $\int_{0}^{1}\frac{2x}{x^2+4}dx$

[Sol] Let $x^2+4=t$. Since $x^2=t-4$, $2xdx=dt$

x	$0 \longrightarrow 1$
t	$4 \longrightarrow 5$

$$\therefore \int_{0}^{1}\frac{2x}{x^2+4}dx=\int_{4}^{5}\frac{1}{t}dt=\left[\ln|t|\right]_{4}^{5}=\ln\frac{5}{4}$$

(3) $\int_{0}^{1}\frac{x^3}{1+x^2}dx$

[Sol] Let $1+x^2=t$. Since $x^2=t-1$, $2xdx=dt$

x	$0 \longrightarrow 1$
t	$1 \longrightarrow 2$

$$\therefore \int_{0}^{1}\frac{x^3}{1+x^2}dx=\int_{1}^{2}\frac{t-1}{t}\cdot\frac{1}{2}dt$$

$$=\frac{1}{2}\int_{1}^{2}\left(1-\frac{1}{t}\right)dt$$

$$=\frac{1}{2}\left[t-\ln|t|\right]_{1}^{2}$$

$$=\frac{1}{2}(1-\ln2)$$

Integration by Substitution for Definite Integrals

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Definite integrals have the following property.

Property of Definite Integrals II

$$\int_b^a f(x)dx=-\int_a^b f(x)dx$$

Find the following definite integrals using Integration by Substitution.

Ex. $\displaystyle\int_0^1 x\sqrt{1-x}\,dx$

[Sol] Let $\sqrt{1-x}=t$. Since $1-x=t^2$, $x=1-t^2$, $dx=-2t\,dt$

$$\therefore \int_0^1 x\sqrt{1-x}\,dx=\int_1^0(1-t^2)t\cdot(-2t)dt$$

x	$0 \longrightarrow 1$
t	$1 \longrightarrow 0$

$$=2\int_0^1(t^2-t^4)dt$$

$$\boxed{\int_b^a f(x)dx=-\int_a^b f(x)dx}$$

$$=2\left[\frac{1}{3}t^3-\frac{1}{5}t^5\right]_0^1$$

$$=\frac{4}{15}$$

(1) $\displaystyle\int_1^2 x\sqrt{2-x}\,dx$

[Sol] Let $\sqrt{2-x}=t$. Since $2-x=t^2$, $x=2-t^2$, $dx=-2t\,dt$

$$\therefore \int_1^2 x\sqrt{2-x}\,dx=\int_1^0(2-t^2)t\cdot(-2t)dt$$

x	$1 \longrightarrow 2$
t	$1 \longrightarrow 0$

$$=2\int_0^1(2t^2-t^4)dt$$

$$=2\left[\frac{2}{3}t^3-\frac{1}{5}t^5\right]_0^1$$

$$=\frac{14}{15}$$

Alternative Solution

Let $2-x=t$. Since $x=2-t$, $dx=-dt$

x	$1 \longrightarrow 2$
t	$1 \longrightarrow 0$

$$\therefore \int_1^2 x\sqrt{2-x}\,dx=\int_1^0(2-t)\sqrt{t}\cdot(-1)dt=\int_0^1\left(2t^{\frac{1}{2}}-t^{\frac{3}{2}}\right)dt=\left[\frac{4}{3}t^{\frac{3}{2}}-\frac{2}{5}t^{\frac{5}{2}}\right]_0^1=\frac{14}{15}$$

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(2) $\int_0^2 x\sqrt{4-x^2}\,dx$

[Sol] Let $\sqrt{4-x^2}=t$. Since $4-x^2=t^2$, $-2xdx=2tdt$

x	$0 \longrightarrow 2$
t	$2 \longrightarrow 0$

$\therefore\ xdx=-tdt$

$$\therefore\ \int_0^2 x\sqrt{4-x^2}\,dx=\int_2^0 t\cdot(-t)dt$$
$$=\int_0^2 t^2 dt$$
$$=\left[\frac{1}{3}t^3\right]_0^2$$
$$=\mathbf{\frac{8}{3}}$$

Alternative Solution

Let $4-x^2=t$. $-2xdx=dt$

x	$0 \longrightarrow 2$
t	$4 \longrightarrow 0$

$$\therefore\ \int_0^2 x\sqrt{4-x^2}\,dx=\int_4^0 \sqrt{t}\cdot\left(-\frac{1}{2}\right)dt=\frac{1}{2}\int_0^4 t^{\frac{1}{2}}dt=\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\right]_0^4=\frac{8}{3}$$

(3) $\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx$

[Sol] Let $\sqrt{1-x^2}=t$. Since $1-x^2=t^2$, $-2xdx=2tdt$

x	$0 \longrightarrow \frac{1}{2}$
t	$1 \longrightarrow \frac{\sqrt{3}}{2}$

$\therefore\ xdx=-tdt$

$$\therefore\ \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx=\int_1^{\frac{\sqrt{3}}{2}} \frac{1}{t}\cdot(-t)dt$$
$$=\int_{\frac{\sqrt{3}}{2}}^1 dt$$
$$=\Big[\,t\,\Big]_{\frac{\sqrt{3}}{2}}^1$$
$$=\mathbf{1-\frac{\sqrt{3}}{2}}$$

Alternative Solution

Let $1-x^2=t$. $-2xdx=dt$

x	$0 \longrightarrow \frac{1}{2}$
t	$1 \longrightarrow \frac{3}{4}$

$$\therefore\ \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx=\int_1^{\frac{3}{4}} \frac{1}{\sqrt{t}}\cdot\left(-\frac{1}{2}\right)dt=\frac{1}{2}\int_{\frac{3}{4}}^1 t^{-\frac{1}{2}}dt=\frac{1}{2}\left[2t^{\frac{1}{2}}\right]_{\frac{3}{4}}^1=\mathbf{1-\frac{\sqrt{3}}{2}}$$

Integration by Substitution for Definite Integrals

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Find the following definite integrals using Integration by Substitution.

Ex. $\int_1^e \frac{\ln x}{x}\,dx$

[Sol] Let $\ln x = t$. $\frac{1}{x}dx = dt$

x	$1 \longrightarrow e$
t	$0 \longrightarrow 1$

$$\therefore \int_1^e \frac{\ln x}{x}\,dx = \int_0^1 t\,dt$$
$$= \left[\frac{1}{2}t^2\right]_0^1$$
$$= \frac{1}{2}$$

(1) $\int_e^{e^2} \frac{dx}{x\ln x}$

[Sol] Let $\ln x = t$. $\frac{1}{x}dx = dt$

x	$e \longrightarrow e^2$
t	$1 \longrightarrow 2$

$$\therefore \int_e^{e^2} \frac{dx}{x\ln x} = \int_1^2 \frac{1}{t}\,dt$$
$$= \left[\ln|t|\right]_1^2$$
$$= \mathbf{\ln 2}$$

(2) $\int_0^1 \frac{x}{1+x^2}\ln(1+x^2)\,dx$

[Sol] Let $\ln(1+x^2) = t$. $\frac{2x}{1+x^2}dx = dt$

x	$0 \longrightarrow 1$
t	$0 \longrightarrow \ln 2$

$$\therefore \int_0^1 \frac{x}{1+x^2}\ln(1+x^2)\,dx = \int_0^{\ln 2} t\cdot\frac{1}{2}\,dt$$
$$= \frac{1}{2}\int_0^{\ln 2} t\,dt$$
$$= \frac{1}{2}\left[\frac{1}{2}t^2\right]_0^{\ln 2}$$
$$= \mathbf{\frac{1}{4}(\ln 2)^2}$$

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Ex.

$$\int_0^1 \frac{e^x}{e^x+1}\,dx$$

[Sol] Let $e^x+1=t$. $e^x dx=dt$

x	$0 \longrightarrow 1$
t	$2 \longrightarrow e+1$

$$\therefore \int_0^1 \frac{e^x}{e^x+1}\,dx=\int_2^{e+1}\frac{dt}{t}$$

$$=\Big[\ln|t|\Big]_2^{e+1}$$

$$=\ln\frac{e+1}{2}$$

$\ln M-\ln N=\ln\frac{M}{N}$

Since $\frac{e+1}{2}>0$, $\left|\frac{e+1}{2}\right|=\frac{e+1}{2}$

(3) $\displaystyle\int_0^1 \frac{e^x}{(e^x+1)^2}\,dx$

[Sol] Let $e^x+1=t$. $e^x dx=dt$

x	$0 \longrightarrow 1$
t	$2 \longrightarrow e+1$

$$\therefore \int_0^1 \frac{e^x}{(e^x+1)^2}\,dx=\int_2^{e+1}\frac{dt}{t^2}$$

$$=\left[-\frac{1}{t}\right]_2^{e+1}$$

$$=-\frac{1}{e+1}+\frac{1}{2}\quad\left[=\frac{e-1}{2(e+1)}\right]$$

(4) $\displaystyle\int_0^2 \frac{4e^x+8}{e^x+2x}\,dx$

[Sol] Let $e^x+2x=t$. $(e^x+2)dx=dt$

x	$0 \longrightarrow 2$
t	$1 \longrightarrow e^2+4$

$$\therefore \int_0^2 \frac{4e^x+8}{e^x+2x}\,dx=\int_0^2\frac{4(e^x+2)}{e^x+2x}\,dx$$

$$=\int_1^{e^2+4}\frac{4}{t}\,dt$$

$$=4\Big[\ln|t|\Big]_1^{e^2+4}$$

$$=4\ln(e^2+4)$$

Since $e^2+4>0$, $|e^2+4|=e^2+4$

Integration by Substitution for Definite Integrals

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Find the following definite integrals using Integration by Substitution.

Ex. $\displaystyle\int_0^{\frac{\pi}{6}} \sin^2 x \cos x\, dx$

[Sol] Let $\sin x = t$. $\cos x\, dx = dt$

x	$0 \longrightarrow \frac{\pi}{6}$
t	$0 \longrightarrow \frac{1}{2}$

$$\therefore \int_0^{\frac{\pi}{6}} \sin^2 x \cos x\, dx = \int_0^{\frac{1}{2}} t^2 dt$$

$$= \left[\frac{1}{3}t^3\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{24}$$

(1) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^5 x \cos x\, dx$

[Sol] Let $\sin x = t$. $\cos x\, dx = dt$

x	$0 \longrightarrow \frac{\pi}{4}$
t	$0 \longrightarrow \frac{\sqrt{2}}{2}$

$$\therefore \int_0^{\frac{\pi}{4}} \sin^5 x \cos x\, dx = \int_0^{\frac{\sqrt{2}}{2}} t^5 dt$$

$$= \left[\frac{1}{6}t^6\right]_0^{\frac{\sqrt{2}}{2}}$$

$$= \frac{1}{48}$$

(2) $\displaystyle\int_0^{\frac{\pi}{2}} (1+\cos^2 x) \sin x\, dx$

[Sol] Let $\cos x = t$. $-\sin x\, dx = dt$

x	$0 \longrightarrow \frac{\pi}{2}$
t	$1 \longrightarrow 0$

$$\therefore \int_0^{\frac{\pi}{2}} (1+\cos^2 x) \sin x\, dx = \int_1^0 (1+t^2)\cdot(-1)dt$$

$$= \int_0^1 (1+t^2)dt$$

$$= \left[t + \frac{1}{3}t^3\right]_0^1$$

$$= \frac{4}{3}$$

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(3) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^3 x\,dx$

[Sol] $\displaystyle\int_0^{\frac{\pi}{4}} \sin^3 x\,dx = \int_0^{\frac{\pi}{4}} (1-\cos^2 x)\sin x\,dx$

Let $\cos x = t$. $-\sin x\,dx = dt$

$$\begin{array}{c|c} x & 0 \longrightarrow \frac{\pi}{4} \\ \hline t & 1 \longrightarrow \frac{\sqrt{2}}{2} \end{array}$$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{4}} \sin^3 x\,dx &= \int_1^{\frac{\sqrt{2}}{2}} (1-t^2)\cdot(-1)dt \\ &= \int_{\frac{\sqrt{2}}{2}}^1 (1-t^2)dt \\ &= \left[t-\frac{1}{3}t^3\right]_{\frac{\sqrt{2}}{2}}^1 \\ &= \mathbf{\frac{2}{3}-\frac{5\sqrt{2}}{12}} \quad \left[=\mathbf{\frac{1}{12}(8-5\sqrt{2})}\right] \end{aligned}$$

(4) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{1+\cos x}\,dx$

[Sol] $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{1+\cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{(1-\cos^2 x)\sin x}{1+\cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{(1+\cos x)(1-\cos x)\sin x}{1+\cos x}\,dx$

$$= \int_0^{\frac{\pi}{2}} (1-\cos x)\sin x\,dx$$

Let $\cos x = t$. $-\sin x\,dx = dt$

$$\begin{array}{c|c} x & 0 \longrightarrow \frac{\pi}{2} \\ \hline t & 1 \longrightarrow 0 \end{array}$$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{1+\cos x}\,dx &= \int_1^0 (1-t)\cdot(-1)dt \\ &= \int_0^1 (1-t)dt \\ &= \left[t-\frac{1}{2}t^2\right]_0^1 \\ &= \mathbf{\frac{1}{2}} \end{aligned}$$

[Alternative Solution

$\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{1+\cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{(1-\cos^2 x)\sin x}{1+\cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{(1+\cos x)(1-\cos x)\sin x}{1+\cos x}\,dx = \int_0^{\frac{\pi}{2}} (1-\cos x)\sin x\,dx$

Let $1-\cos x = t$. $\sin x\,dx = dt$

$$\begin{array}{c|c} x & 0 \longrightarrow \frac{\pi}{2} \\ \hline t & 0 \longrightarrow 1 \end{array}$$

$\therefore \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{1+\cos x}\,dx = \int_0^1 t\,dt = \left[\frac{1}{2}t^2\right]_0^1 = \mathbf{\frac{1}{2}}$]

Integration by Substitution for Definite Integrals

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Find the following definite integrals using Integration by Substitution.

Ex. $\int_0^3 \sqrt{9-x^2}\,dx$

[Sol] Let $x=3\sin\theta$. $dx=3\cos\theta\,d\theta$

When $0 \leqslant \theta \leqslant \frac{\pi}{2}$, $\cos\theta \geqslant 0$; therefore,

$\sqrt{9-x^2}=\sqrt{9(1-\sin^2\theta)}=3\cos\theta$

$$\therefore \int_0^3 \sqrt{9-x^2}\,dx=\int_0^{\frac{\pi}{2}} 3\cos\theta\cdot 3\cos\theta\,d\theta$$
$$=9\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$
$$=\frac{9}{2}\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta$$
$$=\frac{9}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$
$$=\frac{9}{4}\pi$$

x	$0 \longrightarrow 3$
θ	$0 \longrightarrow \frac{\pi}{2}$

When $x=0$, $\sin\theta=0$
When $x=3$, $\sin\theta=1$

$\cos^2\alpha=\frac{1+\cos 2\alpha}{2}$

(1) $\int_0^a \sqrt{a^2-x^2}\,dx \quad (a>0)$

[Sol] Let $x=a\sin\theta$. $dx=a\cos\theta\,d\theta$

When $0 \leqslant \theta \leqslant \frac{\pi}{2}$, $\cos\theta \geqslant 0$; therefore,

$\sqrt{a^2-x^2}=\sqrt{a^2(1-\sin^2\theta)}=a\cos\theta$

$$\therefore \int_0^a \sqrt{a^2-x^2}\,dx=\int_0^{\frac{\pi}{2}} a\cos\theta\cdot a\cos\theta\,d\theta$$
$$=a^2\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$
$$=\frac{1}{2}a^2\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta$$
$$=\frac{1}{2}a^2\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$
$$=\frac{1}{4}\pi a^2$$

x	$0 \longrightarrow a$
θ	$0 \longrightarrow \frac{\pi}{2}$

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O105b

(2) $\int_{-1}^{\sqrt{3}} \sqrt{4-x^2}\,dx$

[Sol] Let $x=2\sin\theta$. $dx=2\cos\theta\,d\theta$

x	$-1 \longrightarrow \sqrt{3}$
θ	$-\frac{\pi}{6} \longrightarrow \frac{\pi}{3}$

When $-\frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{3}$, $\cos\theta > 0$; therefore,

$$\sqrt{4-x^2}=\sqrt{4(1-\sin^2\theta)}=2\cos\theta$$

$$\therefore \int_{-1}^{\sqrt{3}} \sqrt{4-x^2}\,dx=\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} 2\cos\theta\cdot 2\cos\theta\,d\theta$$

$$=4\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2\theta\,d\theta$$

$$=2\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} (1+\cos 2\theta)\,d\theta$$

$$=2\left[\theta+\frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$=\pi+\sqrt{3}$$

(3) $\int_{0}^{2} \frac{dx}{\sqrt{16-x^2}}$

[Sol] Let $x=4\sin\theta$. $dx=4\cos\theta\,d\theta$

x	$0 \longrightarrow 2$
θ	$0 \longrightarrow \frac{\pi}{6}$

When $0 \leqslant \theta \leqslant \frac{\pi}{6}$, $\cos\theta > 0$; therefore,

$$\sqrt{16-x^2}=\sqrt{16(1-\sin^2\theta)}=4\cos\theta$$

$$\therefore \int_{0}^{2} \frac{dx}{\sqrt{16-x^2}}=\int_{0}^{\frac{\pi}{6}} \frac{1}{4\cos\theta}\cdot 4\cos\theta\,d\theta$$

$$=\int_{0}^{\frac{\pi}{6}} d\theta$$

$$=\Big[\theta\Big]_{0}^{\frac{\pi}{6}}$$

$$=\frac{\pi}{6}$$

In the case of **Ex.**, when $x=0$ and $x=3$, values other than $\theta=0$ and $\theta=\frac{\pi}{2}$ also exist, but it is

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Find the following definite integrals using Integration by Substitution.

Ex. $\displaystyle\int_{-2}^{2}\frac{dx}{x^2+4}$

When $x=-2$, $\tan\theta=-1$
When $x=2$, $\tan\theta=1$

[Sol] Let $x=2\tan\theta$. $dx=\dfrac{2}{\cos^2\theta}d\theta$

x	$-2 \longrightarrow 2$
θ	$-\frac{\pi}{4} \longrightarrow \frac{\pi}{4}$

$$x^2+4=4(\tan^2\theta+1)=\frac{4}{\cos^2\theta}$$

$1+\tan^2\theta=\dfrac{1}{\cos^2\theta}$

$$\therefore \int_{-2}^{2}\frac{dx}{x^2+4}=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{\frac{4}{\cos^2\theta}}\cdot\frac{2}{\cos^2\theta}d\theta$$

$$=\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}d\theta$$

$$=\frac{1}{2}\Big[\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=\frac{\pi}{4}$$

(1) $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}}\frac{dx}{x^2+9}$

[Sol] Let $x=3\tan\theta$. $dx=\dfrac{3}{\cos^2\theta}d\theta$

x	$-\sqrt{3} \longrightarrow \sqrt{3}$
θ	$-\frac{\pi}{6} \longrightarrow \frac{\pi}{6}$

$$x^2+9=9(\tan^2\theta+1)=\frac{9}{\cos^2\theta}$$

$$\therefore \int_{-\sqrt{3}}^{\sqrt{3}}\frac{dx}{x^2+9}=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\frac{1}{\frac{9}{\cos^2\theta}}\cdot\frac{3}{\cos^2\theta}d\theta$$

$$=\frac{1}{3}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}d\theta$$

$$=\frac{1}{3}\Big[\theta\Big]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}=\frac{\pi}{9}$$

In the case of **Ex.**, when $x=2$, $\theta=\frac{5}{4}\pi$ also exists. However, $x=\tan\theta$ cannot be defined by $\theta=\frac{\pi}{2}$. Therefore, θ corresponding to x should not be stated as $-\frac{\pi}{4}\longrightarrow\frac{5}{4}\pi$.

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(2) $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{3+x^2}$

x	$1 \longrightarrow \sqrt{3}$
θ	$\frac{\pi}{6} \longrightarrow \frac{\pi}{4}$

[Sol] Let $x=\sqrt{3}\tan\theta$. $dx=\dfrac{\sqrt{3}}{\cos^2\theta}\,d\theta$

$$3+x^2=3(1+\tan^2\theta)=\frac{3}{\cos^2\theta}$$

$$\therefore \int_1^{\sqrt{3}} \frac{dx}{3+x^2}=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\frac{3}{\cos^2\theta}}\cdot\frac{\sqrt{3}}{\cos^2\theta}\,d\theta$$

$$=\frac{\sqrt{3}}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} d\theta$$

$$=\frac{\sqrt{3}}{3}\Big[\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{4}}=\frac{\sqrt{3}}{36}\pi$$

(3) $\displaystyle\int_0^{1} \frac{dx}{(1+x^2)^2}$

x	$0 \longrightarrow 1$
θ	$0 \longrightarrow \frac{\pi}{4}$

[Sol] Let $x=\tan\theta$. $dx=\dfrac{d\theta}{\cos^2\theta}$

$$(1+x^2)^2=(1+\tan^2\theta)^2=\frac{1}{\cos^4\theta}$$

$$\therefore \int_0^{1} \frac{dx}{(1+x^2)^2}=\int_0^{\frac{\pi}{4}} \frac{1}{\frac{1}{\cos^4\theta}}\cdot\frac{d\theta}{\cos^2\theta}$$

$$=\int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta$$

$\cos^2\alpha=\dfrac{1+\cos 2\alpha}{2}$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}(1+\cos 2\theta)\,d\theta$$

$$=\frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}=\frac{\pi}{8}+\frac{1}{4}\quad\left[=\frac{1}{8}(\pi+2)\right]$$

Note Summary

Let $x=a\sin\theta$ for the definite integral of $\sqrt{a^2-x^2}$.

Let $x=a\tan\theta$ for the definite integral of $\dfrac{1}{x^2+a^2}$.

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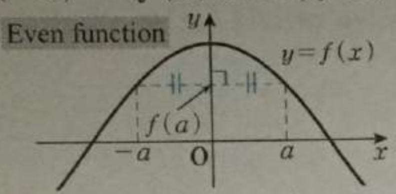
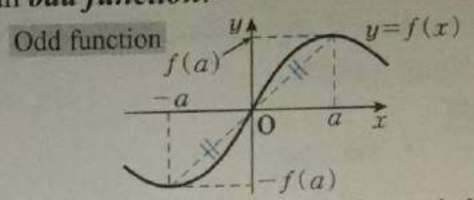
For the function $f(x)$,

when $f(-x)=f(x)$ is true, $f(x)$ is an ***even function***; and

when $f(-x)=-f(x)$ is true, $f(x)$ is an ***odd function***.





The graph of an even function is symmetric with respect to the y-axis, and the graph of an odd function is symmetric with respect to the origin.

Prove the following two identities for the definite integrals of even and odd functions.

[1] When $f(x)$ is an even function, $\int_{-a}^{a} f(x)dx=2\int_{0}^{a} f(x)dx$

[2] When $f(x)$ is an odd function, $\int_{-a}^{a} f(x)dx=0$

[Sol] $\int_{-a}^{a} f(x)dx=\int_{-a}^{0} f(x)dx+\int_{0}^{a} f(x)dx$ ⟵ $\int_{a}^{b} f(x)dx=\int_{a}^{c} f(x)dx+\int_{c}^{b} f(x)dx$

For $\int_{-a}^{0} f(x)dx$, let $x=-t$. $dx=-dt$

x	$-a \longrightarrow 0$
t	$a \longrightarrow 0$

$\therefore \int_{-a}^{0} f(x)dx=\int_{a}^{0} f(-t)\cdot(-1)dt=\int_{0}^{a} f(-t)dt$ ⋯①

The values of definite integrals are not affected by the variable of integration.

[1] Since $f(x)$ is an even function, $f(-x)=\boxed{f(x)}$

From ①, $\int_{-a}^{0} f(x)dx=\int_{0}^{a} f(-t)dt=\int_{0}^{a}\boxed{f(t)}\,dt=\int_{0}^{a} f(x)dx$

$\therefore \int_{-a}^{a} f(x)dx=\int_{-a}^{0} f(x)dx+\int_{0}^{a} f(x)dx=2\int_{0}^{a} f(x)dx$

[2] Since $f(x)$ is an odd function, $f(-x)=\boxed{-f(x)}$

From ①, $\int_{-a}^{0} f(x)dx=\int_{0}^{a} f(-t)dt=\int_{0}^{a}[\boxed{-f(t)}]dt=-\int_{0}^{a} f(x)dx$

$\therefore \int_{-a}^{a} f(x)dx=\int_{-a}^{0} f(x)dx+\int_{0}^{a} f(x)dx=0$

Answers: $f(x)$, $f(t)$, $-f(x)$, $-f(t)$

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Integration of Even/Odd Functions

When $f(x)$ is an even function, $\int_{-a}^{a} f(x)dx=2\int_{0}^{a} f(x)dx$

When $f(x)$ is an odd function, $\int_{-a}^{a} f(x)dx=0$

Determine if each given function is an even or odd function and then find the definite integral.

Ex.

$\int_{-\pi}^{\pi} \sin x\,dx$

[Sol] Let $f(x)=\sin x$. $f(-x)=\sin(-x)=-\sin x=-f(x)$

Therefore, $f(x)$ is an odd function.

$\sin(-\theta)=-\sin\theta$

$\therefore \int_{-\pi}^{\pi} \sin x\,dx=0$

(1) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$

[Sol] Let $f(x)=\cos x$. $f(-x)=\cos(-x)=\cos x=f(x)$

Therefore, $f(x)$ is an even function.

$\cos(-\theta)=\cos\theta$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx=2\int_{0}^{\frac{\pi}{2}} \cos x\,dx$$

$$=2\Big[\sin x\Big]_0^{\frac{\pi}{2}}$$

$$=\mathbf{2}$$

(2) $\int_{-e}^{e} xe^{x^2}dx$

[Sol] Let $f(x)=xe^{x^2}$. $f(-x)=-xe^{(-x)^2}=-xe^{x^2}=-f(x)$

Therefore, $f(x)$ is an odd function.

$\therefore \int_{-e}^{e} xe^{x^2}dx=\mathbf{0}$

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Determine if each given function is an even or odd function and then find the definite integral.

(1) $\int_{-a}^{a} x\sqrt{a^2-x^2}\,dx$

[Sol] Let $f(x)=x\sqrt{a^2-x^2}$.

$$f(-x)=-x\sqrt{a^2-(-x)^2}=-x\sqrt{a^2-x^2}=-f(x)$$

Therefore, $f(x)$ is an odd function.

$$\therefore\ \int_{-a}^{a} x\sqrt{a^2-x^2}\,dx=0$$

(2) $\int_{-a}^{a}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)dx$

[Sol] Let $f(x)=e^{\frac{x}{a}}+e^{-\frac{x}{a}}$.

$$f(-x)=e^{\frac{-x}{a}}+e^{-\frac{-x}{a}}=e^{-\frac{x}{a}}+e^{\frac{x}{a}}=f(x)$$

Therefore, $f(x)$ is an even function.

$$\therefore\ \int_{-a}^{a}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)dx=2\int_{0}^{a}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)dx$$

$$=2\left[ae^{\frac{x}{a}}-ae^{-\frac{x}{a}}\right]_0^a$$

$$=2a\left(e-\frac{1}{e}\right)$$

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(3) $\int_{-\pi}^{\pi} \cos x \sin^3 x \, dx$

[Sol] Let $f(x)=\cos x \sin^3 x$.

$$f(-x)=\cos(-x)\sin^3(-x)=-\cos x \sin^3 x=-f(x)$$

Therefore, $f(x)$ is an odd function.

$$\therefore \int_{-\pi}^{\pi} \cos x \sin^3 x \, dx = \mathbf{0}$$

(4) $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (\cos x + x^2 \sin x) dx$

[Sol] Since $\cos x$ is an $\boxed{\textbf{even}}$ function and $x^2 \sin x$ is an $\boxed{\textbf{odd}}$ function,

$$\begin{aligned}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (\cos x + x^2 \sin x) dx &= 2\int_0^{\frac{\pi}{3}} \boxed{\boldsymbol{\cos x}}\, dx \\ &= 2\Big[\sin x\Big]_0^{\frac{\pi}{3}} \\ &= \sqrt{3}\end{aligned}$$

(5) $\int_{-\pi}^{\pi} \sin x(\sin x + \cos x) dx$

[Sol] $\int_{-\pi}^{\pi} \sin x(\sin x+\cos x)dx = \int_{-\pi}^{\pi}(\sin^2 x + \sin x \cos x)dx$

Since $\sin^2 x$ is an even function and $\sin x \cos x$ is an odd function,

$$\begin{aligned}\int_{-\pi}^{\pi} \sin x(\sin x + \cos x)dx &= 2\int_0^{\pi} \sin^2 x \, dx \\ &= \int_0^{\pi} (1-\cos 2x) dx \\ &= \left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} \\ &= \pi\end{aligned}$$

$\sin^2\alpha = \dfrac{1-\cos 2\alpha}{2}$

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1. Prove the following identities.

Ex. $\int_0^1 f(x)dx=\int_0^1 f(1-x)dx$

[Sol] Let $1-x=t$. Since $x=1-t$, $dx=-dt$

x	$0 \longrightarrow 1$
t	$1 \longrightarrow 0$

$$\therefore \int_0^1 f(x)dx=\int_1^0 f(1-t)\cdot(-1)dt$$
$$=\int_0^1 f(1-t)dt$$
$$=\int_0^1 f(1-x)dx$$

The values of definite integrals are not affected by the variable of integration.
$\int_a^b f(t)dt=\int_a^b f(x)dx$

(1) $\int_2^5 f(x)dx=\int_2^5 f(7-x)dx$

[Sol] Let $7-x=t$. Since $x=7-t$, $dx=-dt$

x	$2 \longrightarrow 5$
t	$5 \longrightarrow 2$

$$\therefore \int_2^5 f(x)dx=\int_5^2 f(7-t)\cdot(-1)dt$$
$$=\int_2^5 f(7-t)dt$$
$$=\int_2^5 f(7-x)dx$$

(2) $\int_0^1 [f(x)+f(2-x)]dx=\int_0^2 f(x)dx$

[Sol] $\int_0^1 [f(x)+f(2-x)]dx=\int_0^1 f(x)dx+\int_0^1 f(2-x)dx \quad \cdots ①$

For $\int_0^1 f(2-x)dx$, let $2-x=t$. Since $x=2-t$, $dx=-dt$

x	$0 \longrightarrow 1$
t	$2 \longrightarrow 1$

$$\therefore \int_0^1 f(2-x)dx=\int_2^1 f(t)\cdot(-1)dt$$
$$=\int_1^2 f(t)dt$$
$$=\int_1^2 f(x)dx$$

$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$

Therefore, from ①,

$$\int_0^1 [f(x)+f(2-x)]dx=\int_0^1 f(x)dx+\int_1^2 f(x)dx=\int_0^2 f(x)dx$$

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2. Let $x=\frac{\pi}{2}-t$. Find the value of the following definite integral.

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx$$

[Sol] Since $x=\frac{\pi}{2}-t$, $dx=-dt$

x	$0 \longrightarrow \frac{\pi}{2}$
t	$\frac{\pi}{2} \longrightarrow 0$

$$\sin x=\sin\left(\frac{\pi}{2}-t\right)=\cos t \quad \Leftarrow \quad \sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$$

$$\cos x=\cos\left(\frac{\pi}{2}-t\right)=\sin t \quad \Leftarrow \quad \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx=\int_{\frac{\pi}{2}}^{0} \frac{\cos t}{\cos t+\sin t}\cdot(-1)\,dt$$

$$=\int_0^{\frac{\pi}{2}} \frac{\cos t}{\cos t+\sin t}\,dt$$

$$=\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}\,dx$$

Let $I=\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx=\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}\,dx.$

$$2I=\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx+\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}\,dx$$

$$=\int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\sin x+\cos x}\,dx$$

$$\int_a^b f(x)\,dx+\int_a^b g(x)\,dx=\int_a^b [f(x)+g(x)]\,dx$$

$$=\int_0^{\frac{\pi}{2}} dx$$

$$=\Big[x\Big]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{2}$$

$$\therefore I=\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx=\frac{\pi}{4}$$

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Find the following definite integrals using Integration by Substitution.

➡ O102

(1) $\int_0^1 x\sqrt{1-x^2}\,dx$

[Sol] Let $\sqrt{1-x^2}=t$. Since $1-x^2=t^2$, $-2x\,dx=2t\,dt$

x	$0 \longrightarrow 1$
t	$1 \longrightarrow 0$

$\therefore\ x\,dx=-t\,dt$

$$\therefore\ \int_0^1 x\sqrt{1-x^2}\,dx=\int_1^0 t\cdot(-t)\,dt$$

$$=\int_0^1 t^2\,dt$$

$$=\left[\frac{1}{3}t^3\right]_0^1$$

$$=\frac{1}{3}$$

Alternative Solution

Let $1-x^2=t$. $-2x\,dx=dt$

x	$0 \longrightarrow 1$
t	$1 \longrightarrow 0$

$$\therefore\ \int_0^1 x\sqrt{1-x^2}\,dx=\int_1^0 \sqrt{t}\cdot\left(-\frac{1}{2}\right)dt=\frac{1}{2}\int_0^1 t^{\frac{1}{2}}dt=\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\right]_0^1=\frac{1}{3}$$

➡ O104

(2) $\int_0^{\frac{\pi}{4}} \cos^3 x\,dx$

[Sol] $\int_0^{\frac{\pi}{4}} \cos^3 x\,dx=\int_0^{\frac{\pi}{4}}(1-\sin^2 x)\cos x\,dx$

x	$0 \longrightarrow \frac{\pi}{4}$
t	$0 \longrightarrow \frac{\sqrt{2}}{2}$

Let $\sin x=t$. $\cos x\,dx=dt$

$$\therefore\ \int_0^{\frac{\pi}{4}} \cos^3 x\,dx=\int_0^{\frac{\sqrt{2}}{2}}(1-t^2)\,dt$$

$$=\left[t-\frac{1}{3}t^3\right]_0^{\frac{\sqrt{2}}{2}}$$

$$=\frac{5\sqrt{2}}{12}$$

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(3) $\int_0^1 \sqrt{1-x^2}\,dx$ ➡ 0105

[Sol] Let $x=\sin\theta$. $dx=\cos\theta\,d\theta$

x	$0 \longrightarrow 1$
θ	$0 \longrightarrow \frac{\pi}{2}$

When $0 \leqslant \theta \leqslant \frac{\pi}{2}$, $\cos\theta \geqslant 0$; therefore,

$$\sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\cos\theta$$

$$\begin{aligned}\therefore \int_0^1 \sqrt{1-x^2}\,dx &= \int_0^{\frac{\pi}{2}} \cos\theta \cdot \cos\theta\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta \\ &= \frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4}\end{aligned}$$

(4) $\int_0^{\sqrt{2}} \frac{dx}{2+x^2}$ ➡ 0106

[Sol] Let $x=\sqrt{2}\tan\theta$. $dx=\frac{\sqrt{2}}{\cos^2\theta}\,d\theta$

x	$0 \longrightarrow \sqrt{2}$
θ	$0 \longrightarrow \frac{\pi}{4}$

$$2+x^2=2(1+\tan^2\theta)=\frac{2}{\cos^2\theta}$$

$$\begin{aligned}\therefore \int_0^{\sqrt{2}} \frac{dx}{2+x^2} &= \int_0^{\frac{\pi}{4}} \frac{1}{\frac{2}{\cos^2\theta}} \cdot \frac{\sqrt{2}}{\cos^2\theta}\,d\theta \\ &= \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{4}} d\theta \\ &= \frac{\sqrt{2}}{2}\Big[\theta\Big]_0^{\frac{\pi}{4}} \\ &= \frac{\sqrt{2}}{8}\pi\end{aligned}$$

Integration by Parts for Definite Integrals and Functions Represented by Definite Integrals

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As with indefinite integrals, Integration by Parts is also true for definite integrals.

Integration by Parts for Definite Integrals

$$\int_a^b f(x)g'(x)dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)dx$$

This method is called ***Integration by Parts for Definite Integrals***.

Find the following definite integrals using Integration by Parts.

Ex.

$$\int_0^{\frac{\pi}{2}} x\cos x\,dx = \int_0^{\frac{\pi}{2}} x(\sin x)'dx$$

Rearranging into the form $\int_a^b f(x)g'(x)dx$

$$= \Big[x\sin x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (x)'\sin x\,dx$$

$\Big[f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)dx$

$$= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin x\,dx$$

$$= \frac{\pi}{2} - \Big[-\cos x\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - 1$$

(1) $$\int_{\frac{\pi}{2}}^{\pi} x\sin x\,dx = \int_{\frac{\pi}{2}}^{\pi} x(-\cos x)'dx$$

$$= \Big[-x\cos x\Big]_{\frac{\pi}{2}}^{\pi} - \int_{\frac{\pi}{2}}^{\pi} (x)'(-\cos x)dx$$

$$= \pi + \int_{\frac{\pi}{2}}^{\pi} \cos x\,dx$$

$$= \pi + \Big[\sin x\Big]_{\frac{\pi}{2}}^{\pi}$$

$$= \pi - 1$$

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(2) $\int_0^{\frac{\pi}{3}} x\sin 2x\,dx=\int_0^{\frac{\pi}{3}} x\left(-\frac{1}{2}\cos 2x\right)'dx$

$=\left[-\frac{1}{2}x\cos 2x\right]_0^{\frac{\pi}{3}}-\int_0^{\frac{\pi}{3}}(x)'\left(-\frac{1}{2}\cos 2x\right)dx$

$=\frac{\pi}{12}+\frac{1}{2}\int_0^{\frac{\pi}{3}}\cos 2x\,dx$

$=\frac{\pi}{12}+\frac{1}{2}\left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{3}}$

$=\frac{\pi}{12}+\frac{\sqrt{3}}{8}$

(3) $\int_0^{\frac{\pi}{4}}\frac{x}{\cos^2 x}dx=\int_0^{\frac{\pi}{4}}x(\tan x)'dx$

$=\left[x\tan x\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}(x)'\tan x\,dx$

$=\frac{\pi}{4}-\int_0^{\frac{\pi}{4}}\tan x\,dx$

$=\frac{\pi}{4}-\int_0^{\frac{\pi}{4}}\frac{(\cos x)'}{\cos x}\cdot(-1)\,dx$

$\int\frac{g'(x)}{g(x)}dx=\ln|g(x)|+C$

$=\frac{\pi}{4}+\left[\ln|\cos x|\right]_0^{\frac{\pi}{4}}$

$=\frac{\pi}{4}+\ln\frac{\sqrt{2}}{2}\quad\left[=\frac{\pi}{4}-\frac{1}{2}\ln 2\right]\left[=\frac{\pi}{4}-\ln\sqrt{2}\right]$

(4) $\int_\alpha^\beta(x-\alpha)(x-\beta)dx$ (α and β are constants)

[Sol] $\int_\alpha^\beta(x-\alpha)(x-\beta)dx=\int_\alpha^\beta\left[\frac{1}{2}(x-\alpha)^2\right]'(x-\beta)dx$

$=\left[\frac{1}{2}(x-\alpha)^2(x-\beta)\right]_\alpha^\beta-\int_\alpha^\beta\frac{1}{2}(x-\alpha)^2(x-\beta)'dx$

$=-\frac{1}{2}\int_\alpha^\beta(x-\alpha)^2dx$

$=-\frac{1}{2}\left[\frac{1}{3}(x-\alpha)^3\right]_\alpha^\beta$

$=-\frac{1}{6}(\beta-\alpha)^3$

Integration by Parts for Definite Integrals and Functions Represented by Definite Integrals

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Find the following definite integrals using Integration by Parts.

(1) $\int_{-1}^{1} xe^{x}\,dx=\int_{-1}^{1} x(e^{x})'dx$

$=\left[xe^{x}\right]_{-1}^{1}-\int_{-1}^{1}(x)'e^{x}\,dx$

$=e+\frac{1}{e}-\int_{-1}^{1}e^{x}\,dx$

$=e+\frac{1}{e}-\left[e^{x}\right]_{-1}^{1}$

$=\mathbf{\frac{2}{e}}$

(2) $\int_{0}^{1}(2x+3)e^{x}\,dx=\int_{0}^{1}(2x+3)(e^{x})'dx$

$=\left[(2x+3)e^{x}\right]_{0}^{1}-\int_{0}^{1}(2x+3)'e^{x}\,dx$

$=5e-3-2\int_{0}^{1}e^{x}\,dx$

$=5e-3-2\left[e^{x}\right]_{0}^{1}$

$=\mathbf{3e-1}$

(3) $\int_{0}^{1}xe^{-2x}\,dx=\int_{0}^{1}x\left(-\frac{1}{2}e^{-2x}\right)'dx$

$=\left[-\frac{1}{2}xe^{-2x}\right]_{0}^{1}-\int_{0}^{1}(x)'\left(-\frac{1}{2}e^{-2x}\right)dx$

$=-\frac{1}{2e^{2}}+\frac{1}{2}\int_{0}^{1}e^{-2x}\,dx$

$=-\frac{1}{2e^{2}}+\frac{1}{2}\left[-\frac{1}{2}e^{-2x}\right]_{0}^{1}$

$=\mathbf{-\frac{3}{4e^{2}}+\frac{1}{4}}$

O112b

(4) $\displaystyle\int_1^e x\ln x\,dx=\int_1^e\left(\frac{1}{2}x^2\right)'\ln x\,dx$

$\displaystyle=\left[\frac{1}{2}x^2\ln x\right]_1^e-\int_1^e\frac{1}{2}x^2(\ln x)'dx$

$\displaystyle=\frac{1}{2}e^2-\frac{1}{2}\int_1^e x\,dx$

$\displaystyle=\frac{1}{2}e^2-\frac{1}{2}\left[\frac{1}{2}x^2\right]_1^e$

$\displaystyle=\frac{1}{4}e^2+\frac{1}{4}\quad\left[=\frac{1}{4}(e^2+1)\right]$

(5) $\displaystyle\int_1^2\frac{\ln x}{x^2}dx=\int_1^2\left(-\frac{1}{x}\right)'\ln x\,dx$

$\displaystyle=\left[-\frac{1}{x}\ln x\right]_1^2-\int_1^2\left(-\frac{1}{x}\right)(\ln x)'dx$

$\displaystyle=-\frac{1}{2}\ln 2+\int_1^2\frac{dx}{x^2}$

$\displaystyle=-\frac{1}{2}\ln 2+\left[-\frac{1}{x}\right]_1^2$

$\displaystyle=-\frac{1}{2}\ln 2+\frac{1}{2}\quad\left[=\frac{1}{2}(1-\ln 2)\right]$

(6) $\displaystyle\int_e^{2e}\ln x\,dx=\int_e^{2e}(x)'\ln x\,dx$ $\longleftarrow$ $\boxed{\int\ln x\,dx=\int 1\cdot\ln x\,dx}$

$\displaystyle=\left[x\ln x\right]_e^{2e}-\int_e^{2e}x(\ln x)'dx$

$\boxed{\begin{aligned}&\ln 2e\\&=\ln 2+\ln e\\&=\ln 2+1\end{aligned}}$

$\displaystyle=2e\ln 2+e-\int_e^{2e}dx$

$\displaystyle=2e\ln 2+e-\left[x\right]_e^{2e}$

$=2e\ln 2$

KUMON

Integration by Parts for Definite Integrals and Functions Represented by Definite Integrals

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Name

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Time : to :

Find the following definite integrals using Integration by Parts.

Ex.

$$\int_{\frac{\pi}{2}}^{\pi} x^2 \sin x\,dx = \int_{\frac{\pi}{2}}^{\pi} x^2(-\cos x)'dx$$

$$= \Big[-x^2\cos x\Big]_{\frac{\pi}{2}}^{\pi} - \int_{\frac{\pi}{2}}^{\pi}(x^2)'(-\cos x)dx$$

$$= \pi^2 + 2\int_{\frac{\pi}{2}}^{\pi} x\cos x\,dx$$

$$= \pi^2 + 2\int_{\frac{\pi}{2}}^{\pi} x(\sin x)'dx$$

$$= \pi^2 + 2\left\{\Big[x\sin x\Big]_{\frac{\pi}{2}}^{\pi} - \int_{\frac{\pi}{2}}^{\pi}(x)'\sin x\,dx\right\}$$

Using Integration by Parts again

$$= \pi^2 - \pi - 2\int_{\frac{\pi}{2}}^{\pi} \sin x\,dx$$

$$= \pi^2 - \pi - 2\Big[-\cos x\Big]_{\frac{\pi}{2}}^{\pi}$$

$$= \pi^2 - \pi - 2$$

(1) $$\int_0^{\pi} x^2\cos x\,dx = \int_0^{\pi} x^2(\sin x)'dx$$

$$= \Big[x^2\sin x\Big]_0^{\pi} - \int_0^{\pi}(x^2)'\sin x\,dx$$

$$= -2\int_0^{\pi} x\sin x\,dx$$

$$= -2\int_0^{\pi} x(-\cos x)'dx$$

$$= -2\left\{\Big[-x\cos x\Big]_0^{\pi} - \int_0^{\pi}(x)'(-\cos x)dx\right\}$$

$$= -2\pi - 2\int_0^{\pi}\cos x\,dx$$

$$= -2\pi - 2\Big[\sin x\Big]_0^{\pi}$$

$$= -2\pi$$

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0113b

(2) $\int_{-1}^{1} x^2 e^{1-x} dx = \int_{-1}^{1} x^2(-e^{1-x})' dx$

$= \left[-x^2 e^{1-x}\right]_{-1}^{1} - \int_{-1}^{1} (x^2)'(-e^{1-x}) dx$

$= -1 + e^2 + 2\int_{-1}^{1} x e^{1-x} dx$

$= -1 + e^2 + 2\int_{-1}^{1} x(-e^{1-x})' dx$

$= -1 + e^2 + 2\left\{\left[-xe^{1-x}\right]_{-1}^{1} - \int_{-1}^{1} (x)'(-e^{1-x}) dx\right\}$

$= -e^2 - 3 + 2\int_{-1}^{1} e^{1-x} dx$

$= -e^2 - 3 + 2\left[-e^{1-x}\right]_{-1}^{1}$

$= e^2 - 5$

(3) $\int_{1}^{e} (\ln x)^2 dx = \int_{1}^{e} (x)'(\ln x)^2 dx$

$= \left[x(\ln x)^2\right]_{1}^{e} - \int_{1}^{e} x[(\ln x)^2]' dx$

$= e - 2\int_{1}^{e} \ln x \, dx$

$\int x[(\ln x)^2]' dx$

$= \int x \cdot 2\ln x \cdot \frac{1}{x} dx$

$= 2\int \ln x dx$

$= e - 2\int_{1}^{e} (x)' \ln x \, dx$

$= e - 2\left\{\left[x \ln x\right]_{1}^{e} - \int_{1}^{e} x(\ln x)' dx\right\}$

$= -e + 2\int_{1}^{e} dx$

$= -e + 2\left[x\right]_{1}^{e}$

$= e - 2$

Integration by Parts for Definite Integrals and Functions Represented by Definite Integrals

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Name

Date / /

Time : to :

Find the following definite integrals using Integration by Parts.

Ex.

$$\int_0^{\frac{\pi}{2}} e^x \sin x\,dx = \int_0^{\frac{\pi}{2}} (e^x)' \sin x\,dx$$
$$= \Big[e^x \sin x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x (\sin x)' dx$$
$$= e^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x\,dx$$
$$= e^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (e^x)' \cos x\,dx$$
$$= e^{\frac{\pi}{2}} - \left\{\Big[e^x \cos x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x (\cos x)' dx\right\}$$
$$= e^{\frac{\pi}{2}} + 1 - \int_0^{\frac{\pi}{2}} e^x \sin x\,dx$$

Same as the given expression

$$\therefore\ 2\int_0^{\frac{\pi}{2}} e^x \sin x\,dx = e^{\frac{\pi}{2}} + 1$$
$$\therefore\ \int_0^{\frac{\pi}{2}} e^x \sin x\,dx = \frac{1}{2}\left(e^{\frac{\pi}{2}} + 1\right)$$

(1)
$$\int_0^{\frac{\pi}{4}} e^x \cos x\,dx = \int_0^{\frac{\pi}{4}} (e^x)' \cos x\,dx$$
$$= \Big[e^x \cos x\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} e^x (\cos x)' dx$$
$$= \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}} - 1 + \int_0^{\frac{\pi}{4}} e^x \sin x\,dx$$
$$= \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}} - 1 + \int_0^{\frac{\pi}{4}} (e^x)' \sin x\,dx$$
$$= \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}} - 1 + \left\{\Big[e^x \sin x\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} e^x (\sin x)' dx\right\}$$
$$= \sqrt{2} e^{\frac{\pi}{4}} - 1 - \int_0^{\frac{\pi}{4}} e^x \cos x\,dx$$
$$\therefore\ 2\int_0^{\frac{\pi}{4}} e^x \cos x\,dx = \sqrt{2} e^{\frac{\pi}{4}} - 1$$
$$\therefore\ \int_0^{\frac{\pi}{4}} e^x \cos x\,dx = \frac{1}{2}\left(\sqrt{2} e^{\frac{\pi}{4}} - 1\right)$$

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(2) $\displaystyle\int_0^{\pi} e^{-x}\cos x\,dx=\int_0^{\pi}(-e^{-x})'\cos x\,dx$

$\displaystyle=\Big[-e^{-x}\cos x\Big]_0^{\pi}-\int_0^{\pi}(-e^{-x})(\cos x)'dx$

$\displaystyle=\frac{1}{e^{\pi}}+1-\int_0^{\pi}e^{-x}\sin x\,dx$

$\displaystyle=\frac{1}{e^{\pi}}+1-\int_0^{\pi}(-e^{-x})'\sin x\,dx$

$\displaystyle=\frac{1}{e^{\pi}}+1-\left\{\Big[-e^{-x}\sin x\Big]_0^{\pi}-\int_0^{\pi}(-e^{-x})(\sin x)'dx\right\}$

$\displaystyle=\frac{1}{e^{\pi}}+1-\int_0^{\pi}e^{-x}\cos x\,dx$

$\displaystyle\therefore\ 2\int_0^{\pi}e^{-x}\cos x\,dx=\frac{1}{e^{\pi}}+1$

$\displaystyle\therefore\ \int_0^{\pi}e^{-x}\cos x\,dx=\frac{1}{2}\left(\frac{1}{e^{\pi}}+1\right)$

(3) $\displaystyle\int_{\frac{\pi}{2}}^{\pi}e^{2x}\sin 4x\,dx=\int_{\frac{\pi}{2}}^{\pi}\left(\frac{1}{2}e^{2x}\right)'\sin 4x\,dx$

$\displaystyle=\left[\frac{1}{2}e^{2x}\sin 4x\right]_{\frac{\pi}{2}}^{\pi}-\int_{\frac{\pi}{2}}^{\pi}\frac{1}{2}e^{2x}(\sin 4x)'dx$

$\displaystyle=-2\int_{\frac{\pi}{2}}^{\pi}e^{2x}\cos 4x\,dx$

$\displaystyle=-2\int_{\frac{\pi}{2}}^{\pi}\left(\frac{1}{2}e^{2x}\right)'\cos 4x\,dx$

$\displaystyle=-2\left\{\left[\frac{1}{2}e^{2x}\cos 4x\right]_{\frac{\pi}{2}}^{\pi}-\int_{\frac{\pi}{2}}^{\pi}\frac{1}{2}e^{2x}(\cos 4x)'dx\right\}$

$\displaystyle=e^{\pi}-e^{2\pi}-4\int_{\frac{\pi}{2}}^{\pi}e^{2x}\sin 4x\,dx$

$\displaystyle\therefore\ 5\int_{\frac{\pi}{2}}^{\pi}e^{2x}\sin 4x\,dx=e^{\pi}-e^{2\pi}$

$\displaystyle\therefore\ \int_{\frac{\pi}{2}}^{\pi}e^{2x}\sin 4x\,dx=\frac{1}{5}(e^{\pi}-e^{2\pi})\quad\left[=\frac{1}{5}e^{\pi}(1-e^{\pi})\right]$

Integration by Parts for Definite Integrals and Functions Represented by Definite Integrals

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Name

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Time : to :

When a is a constant, $\int_a^x f(t)\,dt$ can be defined by the value of x.

Therefore, it is said to be a function of x.

Let an indefinite integral of $f(t)$ be $F(t)$.

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

Since $F(a)$ is a constant, $[F(a)]' = 0$

Differentiating both sides with respect to x,

$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}[F(x) - F(a)] = F'(x) = f(x)$$

Therefore, the following formula is true.

Definite Integrals and Differentiation

When a is a constant, $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

Differentiate the following functions with respect to x.

Ex. $y = \int_0^x \sqrt{2-t}\,dt$

[Sol] $y' = \sqrt{2-x}$

(1) $y = \int_0^x t\sqrt{t^2+1}\,dt$

[Sol] $y' = x\sqrt{x^2+1}$

(2) $y = \int_2^3 e^t\,dt$

[Sol] Since y is a constant,
$y' = 0$

(3) $y = \int_\pi^x \cos^2\theta\,d\theta$

[Sol] $y' = \cos^2 x$

(4) $y = \int_0^\pi \frac{t\sin t}{8+\sin^2 t}\,dt$

[Sol] Since y is a constant,
$y' = 0$

(5) $y = \int_x^e t\ln t\,dt$

[Sol] $y = -\int_e^x t\ln t\,dt$

$\therefore\ y' = -x\ln x$

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

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0115b

Ex. $y=\int_{x}^{2x}\cos^2 t\,dt$

[Sol] Let $f(t)=\cos^2 t$ and $F'(t)=f(t)$.

$$y=\int_{x}^{2x}f(t)dt=F(2x)-F(x)$$

$$\therefore\ y'=\frac{d}{dx}[F(2x)-F(x)]$$

$$=2F'(2x)-F'(x)$$

$[f(g(x))]'=f'(g(x))g'(x)$

$$=2\cos^2 2x-\cos^2 x$$

(6) $y=\int_{x}^{2x}\sin t\,dt$

[Sol] Let $f(t)=\sin t$ and $F'(t)=f(t)$.

$$y=\int_{x}^{2x}f(t)dt=F(2x)-F(x)$$

$$\therefore\ y'=\frac{d}{dx}[F(2x)-F(x)]$$

$$=2F'(2x)-F'(x)$$

$$=2\sin 2x-\sin x$$

$$\left[=\sin x(4\cos x-1)\right]$$

$\sin 2\alpha=2\sin\alpha\cos\alpha$

(7) $y=\int_{x}^{x^2}e^t\sin t\,dt$

[Sol] Let $f(t)=e^t\sin t$ and $F'(t)=f(t)$.

$$y=\int_{x}^{x^2}f(t)dt=F(x^2)-F(x)$$

$$\therefore\ y'=\frac{d}{dx}[F(x^2)-F(x)]$$

$$=2x\,F'(x^2)-F'(x)$$

$$=2xe^{x^2}\sin x^2-e^x\sin x$$

O116a

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O 116

Integration by Parts for Definite Integrals and Functions Represented by Definite Integrals

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Time : to :

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1. Differentiate the following functions with respect to x.

Ex. $F(x)=\int_0^x (x-t)\cos t\,dt$

x is not affected by the variable of integration t and can therefore be taken to the front of $\int$.

[Sol] $F(x)=x\int_0^x \cos t\,dt-\int_0^x t\cos t\,dt$

$f(x)=x,\ g(x)=\int_0^x \cos t\,dt$
$[f(x)g(x)]'=f'(x)g(x)+f(x)g'(x)$

$$F'(x)=1\cdot\int_0^x \cos t\,dt+x\left(\frac{d}{dx}\int_0^x \cos t\,dt\right)-x\cos x$$

$$=\int_0^x \cos t\,dt+x\cos x-x\cos x$$

$\frac{d}{dx}\int_a^x f(t)dt=f(x)$

$$=\Big[\sin t\Big]_0^x$$

$\frac{d}{dx}\int_a^x f(t)dt=f(x)$

$$=\sin x$$

(1) $F(x)=\int_0^x (x-t)\sin t\,dt$

[Sol] $F(x)=x\int_0^x \sin t\,dt-\int_0^x t\sin t\,dt$

$$F'(x)=1\cdot\int_0^x \sin t\,dt+x\left(\frac{d}{dx}\int_0^x \sin t\,dt\right)-x\sin x$$

$$=\int_0^x \sin t\,dt+x\sin x-x\sin x$$

$$=\Big[-\cos t\Big]_0^x$$

$$=-\cos x+1$$

(2) $F(x)=\int_0^x (x+t)e^t\,dt$

[Sol] $F(x)=x\int_0^x e^t\,dt+\int_0^x te^t\,dt$

$$F'(x)=1\cdot\int_0^x e^t\,dt+x\left(\frac{d}{dx}\int_0^x e^t\,dt\right)+xe^x$$

$$=\int_0^x e^t\,dt+xe^x+xe^x$$

$$=\Big[e^t\Big]_0^x+2xe^x$$

$$=2xe^x+e^x-1\quad\big[=(2x+1)e^x-1\big]$$

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0116b

Definite integrals have the following property.

Property of Definite Integrals III

$$\int_a^a f(x)dx=0$$

2. Given that a is a constant and satisfies the equation
$\int_0^x (x-t)f(t)dt=\cos x-a$ ···①, find the function $f(x)$.
Then, using the above property, find the value of constant a.

[Sol] From ①, $x\int_0^x f(t)dt-\int_0^x tf(t)dt=\cos x-a$

Differentiating both sides with respect to x,

$$1\cdot\int_0^x f(t)dt+x\left[\frac{d}{dx}\int_0^x f(t)dt\right]-xf(x)=-\sin x$$

$$\int_0^x f(t)dt+xf(x)-xf(x)=-\sin x$$

$$\therefore \int_0^x f(t)dt=-\sin x$$

Then, differentiating both sides with respect to x again, $f(x)=-\cos x$

Also, substituting $x=0$ into ①,

$$\int_0^0 (0-t)f(t)dt=\cos 0-a$$

$$0=1-a$$

$\int_a^a f(x)dx=0$

$$\therefore a=1$$

Integration by Parts for Definite Integrals and Functions Represented by Definite Integrals

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Find the function $f(x)$ which satisfies each given equation.

Ex.

$$f(x)=\frac{1}{x}+\int_1^2 tf(t)dt$$

[Sol] Let $\int_1^2 tf(t)dt=k$. $f(x)=\frac{1}{x}+k$ ⟵ $\int_1^2 tf(t)dt$ is a constant.

$$\therefore \int_1^2 tf(t)dt=\int_1^2 t\left(\frac{1}{t}+k\right)dt \quad \longleftarrow \quad f(t)=\frac{1}{t}+k$$

$$=\int_1^2(1+kt)dt$$

$$=\left[t+\frac{1}{2}kt^2\right]_1^2$$

$$=\frac{3}{2}k+1$$

$\int_1^2 tf(t)dt=k$

Therefore, since $k=\frac{3}{2}k+1$, $k=-2$

$$\therefore f(x)=\frac{1}{x}-2$$

(1) $f(x)=\sin x+\int_0^{\frac{\pi}{2}} f(t)dt$

[Sol] Let $\int_0^{\frac{\pi}{2}} f(t)dt=k$. $f(x)=\sin x+k$

$$\therefore \int_0^{\frac{\pi}{2}} f(t)dt=\int_0^{\frac{\pi}{2}}(\sin t+k)dt$$

$$=\left[-\cos t+kt\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\pi k+1$$

Therefore, since $k=\frac{1}{2}\pi k+1$, $k=-\frac{2}{\pi-2}$

$$\therefore f(x)=\sin x-\frac{2}{\pi-2}$$

0117b

(2) $f(x)=\sin x+\int_0^{\frac{\pi}{6}} f(t)\cos t\,dt$

[Sol] Let $\int_0^{\frac{\pi}{6}} f(t)\cos t\,dt=k$. $f(x)=\sin x+k$

$$\begin{aligned}\therefore \int_0^{\frac{\pi}{6}} f(t)\cos t\,dt&=\int_0^{\frac{\pi}{6}}(\sin t+k)\cos t\,dt\\&=\int_0^{\frac{\pi}{6}}(\sin t\cos t+k\cos t)dt\\&=\int_0^{\frac{\pi}{6}}\left(\frac{1}{2}\sin 2t+k\cos t\right)dt\\&=\left[-\frac{1}{4}\cos 2t+k\sin t\right]_0^{\frac{\pi}{6}}\\&=\frac{1}{2}k+\frac{1}{8}\end{aligned}$$

$\sin 2\alpha$
$=2\sin\alpha\cos\alpha$

Therefore, since $k=\frac{1}{2}k+\frac{1}{8}$, $k=\frac{1}{4}$

$\therefore\ f(x)=\sin x+\frac{1}{4}$

(3) $f(x)=x+\int_0^1 f(t)e^{-t}dt$

[Sol] Let $\int_0^1 f(t)e^{-t}dt=k$. $f(x)=x+k$

$$\begin{aligned}\therefore \int_0^1 f(t)e^{-t}dt&=\int_0^1(t+k)e^{-t}dt\\&=\int_0^1(t+k)(-e^{-t})'dt\\&=\left[-(t+k)e^{-t}\right]_0^1-\int_0^1(t+k)'(-e^{-t})dt\\&=-\frac{k+1}{e}+k+\int_0^1 e^{-t}dt\\&=-\frac{k+1}{e}+k+\left[-e^{-t}\right]_0^1\\&=-\frac{k+2}{e}+k+1\end{aligned}$$

Therefore, since $k=-\frac{k+2}{e}+k+1$, $k=e-2$

$\therefore\ f(x)=x+e-2$

O118a

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Integration by Parts for Definite Integrals and Functions Represented by Definite Integrals

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1. Given that a and b are real numbers, find the minimum value of integral $\int_0^1 [\cos \pi x - (ax+b)]^2 dx$ and the corresponding values of a and b.

[Sol] $[\cos \pi x - (ax+b)]^2$

$= \cos^2 \pi x - 2(ax+b)\cos \pi x + (ax+b)^2$

$= \frac{1}{2}\cos 2\pi x - 2(ax+b)\cos \pi x + a^2x^2 + 2abx + b^2 + \frac{1}{2}$

$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$

Then, $\int_0^1 \cos 2\pi x\, dx = \left[\frac{1}{2\pi}\sin 2\pi x\right]_0^1 = 0$

$$\int_0^1 (ax+b)\cos \pi x\, dx = \int_0^1 (ax+b)\left(\frac{1}{\pi}\sin \pi x\right)' dx$$

$$= \left[\frac{1}{\pi}(ax+b)\sin \pi x\right]_0^1 - \int_0^1 (ax+b)' \cdot \frac{1}{\pi}\sin \pi x\, dx$$

$$= -\frac{a}{\pi}\int_0^1 \sin \pi x\, dx$$

$$= -\frac{a}{\pi}\left[-\frac{1}{\pi}\cos \pi x\right]_0^1 = -\frac{2a}{\pi^2}$$

$$\int_0^1 \left(a^2x^2 + 2abx + b^2 + \frac{1}{2}\right)dx = \left[\frac{a^2}{3}x^3 + abx^2 + \left(b^2 + \frac{1}{2}\right)x\right]_0^1$$

$$= \frac{a^2}{3} + ab + b^2 + \frac{1}{2}$$

Therefore,

$$\int_0^1 [\cos \pi x - (ax+b)]^2 dx = \frac{4a}{\pi^2} + \frac{a^2}{3} + ab + b^2 + \frac{1}{2}$$ ※1

$$= \frac{1}{12}\left(a + \frac{24}{\pi^2}\right)^2 + \left(b + \frac{a}{2}\right)^2 - \frac{48}{\pi^4} + \frac{1}{2}$$

Thus, **the minimum value is** $-\frac{48}{\pi^4} + \frac{1}{2}$, **at** $a = -\frac{24}{\pi^2}$, $b = \frac{12}{\pi^2}$.

※2

※1 Since the coefficient of b^2 is 1, it is efficient to first complete the square with respect to b.

※2 The value is a minimum at $a + \frac{24}{\pi^2} = 0$, $b + \frac{a}{2} = 0$.

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O118b

2. Find the relative extreme values of $f(x)=\int_0^x (1-t^2)e^t\,dt.$ ⟵ O13

[Sol] $f(x)=\int_0^x (1-t^2)(e^t)'dt$

$=\left[(1-t^2)e^t\right]_0^x-\int_0^x (1-t^2)'e^t\,dt$

$=(1-x^2)e^x-1+2\int_0^x te^t\,dt$

$=(1-x^2)e^x-1+2\int_0^x t(e^t)'dt$

$=(1-x^2)e^x-1+2\left\{\left[te^t\right]_0^x-\int_0^x (t)'e^t\,dt\right\}$

$=(1-x^2)e^x-1+2xe^x-2\int_0^x e^t\,dt$

$=(1-x^2)e^x-1+2xe^x-2\left[e^t\right]_0^x$

$=-(x-1)^2e^x+1$

Also, $f'(x)=(1-x^2)e^x=(1+x)(1-x)e^x$ ⟵ $\frac{d}{dx}\int_a^x f(t)dt=f(x)$

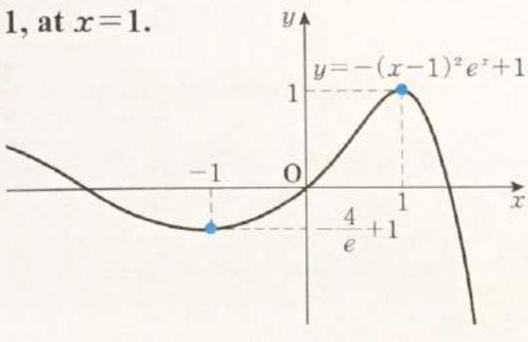
When $f'(x)=0$, $x=\pm1$

x	$\cdots$	-1	$\cdots$	1	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	↘	$-\frac{4}{e}+1$	↗	1	↘

From the variation table,

the relative minimum value is $-\frac{4}{e}+1$, at $x=-1$ and

the relative maximum value is 1, at $x=1$.



Integration by Parts for Definite Integrals and Functions Represented by Definite Integrals

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Name

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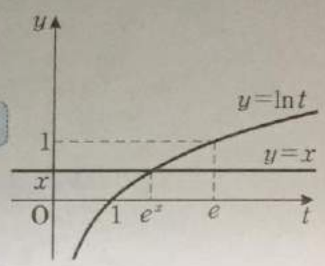
1. Find the minimum value of function $g(x)=\int_1^e |\ln t-x|dt$ and the corresponding value of x when $0 \leqslant x \leqslant 1$. ← O31

[Sol] $|\ln t-x|=\begin{cases} -(\ln t-x) & (1\leqslant t\leqslant e^x) \\ \ln t-x & (e^x\leqslant t\leqslant e) \end{cases}$

When $\ln t-x=0$, $\ln t=x$; therefore, $t=e^x$

Therefore,



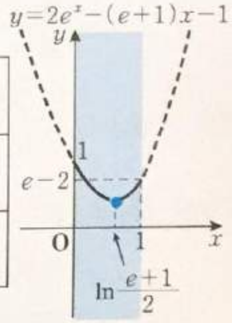
$$g(x)=\int_1^{e^x}[-(\ln t-x)]dt+\int_{e^x}^{e}(\ln t-x)dt$$
$$=\int_1^{e^x}(t)'(-\ln t+x)dt+\int_{e^x}^{e}(t)'(\ln t-x)dt$$
$$=\Big[t(-\ln t+x)\Big]_1^{e^x}-\int_1^{e^x}t(-\ln t+x)'dt$$
$$+\Big[t(\ln t-x)\Big]_{e^x}^{e}-\int_{e^x}^{e}t(\ln t-x)'dt$$
$$=-x+\int_1^{e^x}dt+e(1-x)-\int_{e^x}^{e}dt$$
$$=-(e+1)x+e+\Big[t\Big]_1^{e^x}-\Big[t\Big]_{e^x}^{e}$$
$$=2e^x-(e+1)x-1$$

$$g'(x)=2e^x-(e+1)$$

When $g'(x)=0$ in $0<x<1$, $e^x=\dfrac{e+1}{2}$

$\therefore\ x=\ln\dfrac{e+1}{2}$

x	0	$\cdots$	$\ln\dfrac{e+1}{2}$	$\cdots$	1
$g'(x)$		$-$	0	$+$	
$g(x)$	1	↘	$e-(e+1)\ln\dfrac{e+1}{2}$	↗	$e-2$



From the variation table,

the minimum value is $e-(e+1)\ln\dfrac{e+1}{2}$, at $x=\ln\dfrac{e+1}{2}$.

0119b

2. Solve the following questions regarding definite integral $I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,dx$.

(n is an integer greater than or equal to 0.)

(1) For $n\geqq 2$, find K_n where $I_n=K_nI_{n-2}$.

[Sol] When $n\geqq 2$,

$$I_n=\int_0^{\frac{\pi}{2}}\sin^{n-1}x\cdot\sin x\,dx$$
$$=\int_0^{\frac{\pi}{2}}\sin^{n-1}x(-\cos x)'dx$$
$$=\Big[-\sin^{n-1}x\cos x\Big]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}(\sin^{n-1}x)'(-\cos x)dx$$
$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cos^2x\,dx$$
$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x(1-\sin^2x)dx$$
$$=(n-1)\int_0^{\frac{\pi}{2}}(\sin^{n-2}x-\sin^n x)dx$$
$$=(n-1)(I_{n-2}-I_n)$$

$$(\sin^{n-1}x)'(-\cos x)$$
$$=(n-1)\sin^{n-1-1}x\cdot(\sin x)'\cdot(-\cos x)$$
$$=-(n-1)\sin^{n-2}x\cos^2x$$

Therefore, since $I_n=\frac{n-1}{n}I_{n-2}$, $K_n=\frac{n-1}{n}$

(2) Find $I_5=\int_0^{\frac{\pi}{2}}\sin^5x\,dx$.

[Sol] From (1),

$$I_5=\frac{4}{5}I_3=\frac{4}{5}\cdot\frac{2}{3}I_1=\frac{8}{15}\int_0^{\frac{\pi}{2}}\sin x\,dx$$
$$=\frac{8}{15}\Big[-\cos x\Big]_0^{\frac{\pi}{2}}=\frac{8}{15}$$

(3) Find $I_6=\int_0^{\frac{\pi}{2}}\sin^6x\,dx$.

[Sol] From (1),

$$I_6=\frac{5}{6}I_4=\frac{5}{6}\cdot\frac{3}{4}I_2=\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}I_0$$
$$=\frac{5}{16}\int_0^{\frac{\pi}{2}}dx=\frac{5}{16}\Big[x\Big]_0^{\frac{\pi}{2}}=\frac{5}{32}\pi$$

$$I_0=\int_0^{\frac{\pi}{2}}\sin^0x\,dx=\int_0^{\frac{\pi}{2}}dx$$

O120a

KUMON

Integration by Parts for Definite Integrals and Functions Represented by Definite Integrals

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1. Find the following definite integrals using Integration by Parts.

(1) $\int_0^{\pi} x\cos\frac{x+\pi}{4}\,dx=\int_0^{\pi} x\left(4\sin\frac{x+\pi}{4}\right)'dx$ ➡ O111

$$=\left[4x\sin\frac{x+\pi}{4}\right]_0^{\pi}-\int_0^{\pi}(x)'\cdot 4\sin\frac{x+\pi}{4}\,dx$$

$$=4\pi-4\int_0^{\pi}\sin\frac{x+\pi}{4}\,dx$$

$$=4\pi-4\left[-4\cos\frac{x+\pi}{4}\right]_0^{\pi}$$

$$=\mathbf{4\pi-8\sqrt{2}}$$

(2) $\int_0^{\pi} e^{-x}\sin x\,dx=\int_0^{\pi}(-e^{-x})'\sin x\,dx$ ➡ O114

$$=\left[-e^{-x}\sin x\right]_0^{\pi}-\int_0^{\pi}(-e^{-x})(\sin x)'dx$$

$$=\int_0^{\pi} e^{-x}\cos x\,dx$$

$$=\int_0^{\pi}(-e^{-x})'\cos x\,dx$$

$$=\left[-e^{-x}\cos x\right]_0^{\pi}-\int_0^{\pi}(-e^{-x})(\cos x)'dx$$

$$=\frac{1}{e^{\pi}}+1-\int_0^{\pi} e^{-x}\sin x\,dx$$

$$\therefore\ 2\int_0^{\pi} e^{-x}\sin x\,dx=\frac{1}{e^{\pi}}+1$$

$$\therefore\ \int_0^{\pi} e^{-x}\sin x\,dx=\mathbf{\frac{1}{2}\left(\frac{1}{e^{\pi}}+1\right)}$$

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O120b

2. Differentiate the function $F(x)=\int_0^x (x-t)e^{2t}\,dt$ with respect to x.

➡ O116

[Sol] $F(x)=x\int_0^x e^{2t}\,dt-\int_0^x te^{2t}\,dt$

$$F'(x)=1\cdot\int_0^x e^{2t}\,dt+x\left(\frac{d}{dx}\int_0^x e^{2t}\,dt\right)-xe^{2x}$$

$$=\int_0^x e^{2t}\,dt+xe^{2x}-xe^{2x}$$

$$=\left[\frac{1}{2}e^{2t}\right]_0^x$$

$$=\frac{1}{2}(e^{2x}-1)$$

3. Find the function $f(x)$ which satisfies the following equation.

➡ O117

$$f(x)=\frac{1}{x}+\int_1^3 f(t)\,dt$$

[Sol] Let $\int_1^3 f(t)\,dt=k$. $f(x)=\frac{1}{x}+k$

$$\therefore\ \int_1^3 f(t)\,dt=\int_1^3\left(\frac{1}{t}+k\right)dt$$

$$=\Big[\ln|t|+kt\Big]_1^3$$

$$=2k+\ln 3$$

Therefore, since $k=2k+\ln 3$, $k=-\ln 3$

$$\therefore\ f(x)=\frac{1}{x}-\ln 3$$

O121a

KUMON

Integration by Quadrature and Proof of Inequalities

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Find the area S enclosed by curve $y=x^2$, the x-axis and line $x=1$ using the following method.

[Sol] Divide the interval [0, 1] into n equal subintervals to make n rectangles with a width of $\frac{1}{n}$ and a height the same value of y at the left boundary, as shown in the first graph. Let S_n be the sum of these areas.

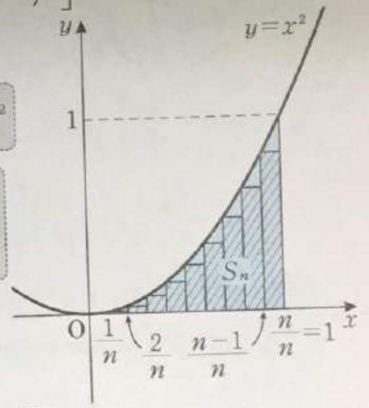
$$S_n=\frac{1}{n}\left[\left(\frac{0}{n}\right)^2+\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\cdots+\left(\frac{n-1}{n}\right)^2\right]$$

$$=\frac{1}{n^3}[0^2+1^2+2^2+\cdots+(n-1)^2]$$

$$=\frac{1}{n^3}\sum_{k=0}^{n-1}k^2 \qquad \sum_{k=0}^{n-1}k^2=\sum_{k=1}^{n-1}k^2$$

$$=\frac{1}{n^3}\cdot\frac{1}{6}(n-1)n(2n-1) \qquad \text{Summation Formula II (N23)}$$

$$=\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) \quad \cdots①$$



Likewise, let T_n be the area of the shaded rectangles in the second graph.

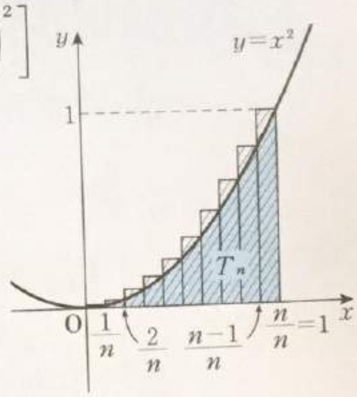
$$T_n=\frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\left(\frac{3}{n}\right)^2+\cdots+\left(\frac{n}{n}\right)^2\right]$$

$$=\frac{1}{n^3}(1^2+2^2+3^2+\cdots+n^2)$$

$$=\frac{1}{n^3}\sum_{k=1}^{\boxed{n}}k^2$$

$$=\frac{1}{n^3}\cdot\frac{1}{6}n(n+1)(\boxed{2n+1})$$

$$=\frac{1}{6}\left(1+\frac{1}{n}\right)\left(\boxed{2+\frac{1}{n}}\right) \quad \cdots②$$



Considering $S_n < S < T_n$,

from ①, $\lim_{n\to\infty}S_n=\frac{1}{6}\cdot1\cdot2=\frac{1}{3}$ and from ②, $\lim_{n\to\infty}T_n=\frac{1}{6}\cdot1\cdot\boxed{2}=\boxed{\frac{1}{3}}$

$\therefore\ S=\boxed{\frac{1}{3}}$

Limits of Sequences and Their Relationships (N67)

Answers: n, $2n+1$, $2+\frac{1}{n}$, 2, $\frac{1}{3}$, $\frac{1}{3}$

0121b

1. In the same way as on side a, find the area S enclosed by curve $y=x^3$, the x-axis and line $x=1$.

[Sol] Divide the interval $[0, 1]$ into n equal subintervals to make n rectangles with a width of $\frac{1}{n}$ and a height the same value of y at the left boundary, as shown in the first graph. Let S_n be the sum of these areas.

$$S_n=\frac{1}{n}\left[\left(\frac{0}{n}\right)^3+\left(\frac{1}{n}\right)^3+\left(\frac{2}{n}\right)^3+\cdots+\left(\frac{n-1}{n}\right)^3\right]$$

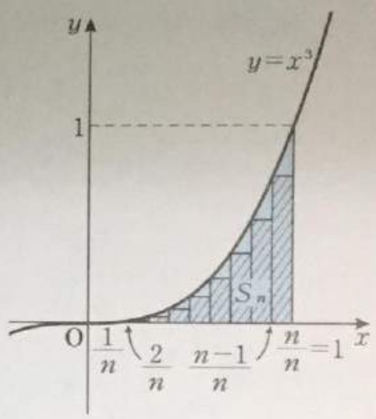
$$=\frac{1}{n^4}[0^3+1^3+2^3+\cdots+(n-1)^3]$$

$$=\frac{1}{n^4}\sum_{k=0}^{n-1}k^3$$

Summation Formula III (N24)

$$=\frac{1}{n^4}\left[\frac{1}{2}(n-1)n\right]^2$$

$$=\frac{1}{4}\left(1-\frac{1}{n}\right)^2 \quad \cdots①$$



Likewise, let T_n be the area of the shaded rectangles in the second graph.

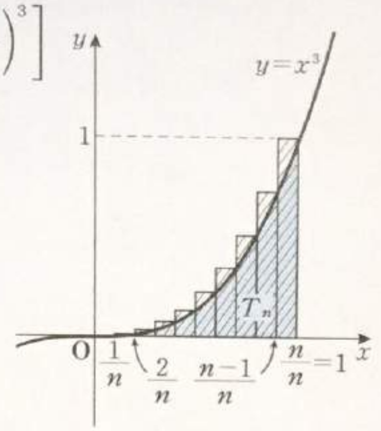
$$T_n=\frac{1}{n}\left[\left(\frac{1}{n}\right)^3+\left(\frac{2}{n}\right)^3+\left(\frac{3}{n}\right)^3+\cdots+\left(\frac{n}{n}\right)^3\right]$$

$$=\frac{1}{n^4}(1^3+2^3+3^3+\cdots+n^3)$$

$$=\frac{1}{n^4}\sum_{k=1}^{n}k^3$$

$$=\frac{1}{n^4}\left[\frac{1}{2}n(n+1)\right]^2$$

$$=\frac{1}{4}\left(1+\frac{1}{n}\right)^2 \quad \cdots②$$



Considering $S_n < S < T_n$,

from ①, $\lim_{n\to\infty} S_n=\frac{1}{4}\cdot 1=\frac{1}{4}$ and from ②, $\lim_{n\to\infty} T_n=\frac{1}{4}\cdot 1=\frac{1}{4}$

$\therefore\ S=\mathbf{\frac{1}{4}}$

O122a

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Integration by Quadrature and Proof of Inequalities

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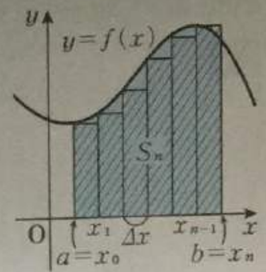
Given that the function $f(x)$ is continuous on the interval $[a, b]$ and $f(x) \geqq 0$ on this interval, let S be the area enclosed by the graph of $y=f(x)$, the x-axis and two lines $x=a$ and $x=b$. Using the conditions above, fill in the following blanks.



[Sol] Dividing the interval $[a, b]$ into n equal subintervals, let both boundaries and dividing points be

$$a=x_0,\ x_1,\ x_2,\ \dots,\ x_{k-1},\ x_k,\ \dots,\ x_{n-1},\ x_n=b$$

Then, let $\Delta x=\dfrac{b-a}{n}$.

The area S_n of the shaded rectangles in the first graph:

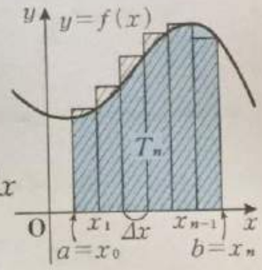
$$S_n=f(x_0)\Delta x+f(x_1)\Delta x+f(x_2)\Delta x+\cdots+f(x_{n-1})\Delta x=\sum_{k=0}^{n-1}\boxed{f(x_k)\Delta x}$$

If n approaches infinity, S_n approaches S.

Using integration, $S=\displaystyle\int_a^b f(x)dx$ ⟵ L141

$$\therefore \lim_{n\to\infty}\sum_{k=0}^{n-1}\boxed{f(x_k)\Delta x}=\int_a^b f(x)dx$$

Also, the same result is obtained by considering the area T_n of the shaded rectangles in the second graph:



$$T_n=f(x_1)\Delta x+f(x_2)\Delta x+\cdots+f(x_{n-1})\Delta x+f(x_n)\Delta x$$

$$=\sum_{k=1}^{\boxed{n}}\boxed{f(x_k)\Delta x}\ ,$$

Therefore, the following equality is true.

$$\lim_{n\to\infty}\sum_{k=1}^{\boxed{n}}\boxed{f(x_k)\Delta x}=\int_a^b f(x)dx$$

Answers: $f(x_k)\Delta x$, $f(x_k)\Delta x$, n, $f(x_k)\Delta x$, n, $f(x_k)\Delta x$

Definite Integrals and Limits of Sums I

If the function $f(x)$ is continuous on the interval $[a, b]$,

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}f(x_k)\Delta x=\lim_{n\to\infty}\sum_{k=1}^{n}f(x_k)\Delta x=\int_a^b f(x)dx$$

where $\Delta x=\dfrac{b-a}{n}$, $x_k=a+k\Delta x$

Even if $f(x)$ is negative, the above equality is true as long as $f(x)$ is continuous on the interval $[a, b]$. The method of finding the area or volume as the limit of the sum by dividing the interval as shown above is called ***Integration by Quadrature***.

0122b

For the equality on side a, let $a=0$ and $b=1$. Then, $\Delta x=\frac{1}{n}$, $x_k=\frac{k}{n}$.
Therefore, the following equality is obtained.

Definite Integrals and Limits of Sums II

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}f\left(\frac{k}{n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)=\int_0^1 f(x)dx$$

Find the following limit values using definite integrals.

Ex.

$$\lim_{n\to\infty}\frac{1}{n\sqrt{n}}(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n})$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\cdots+\sqrt{\frac{n}{n}}\right)$$

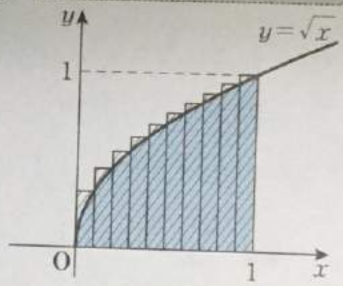
Taking out $\frac{1}{n}$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}$$

Rearranging into the form $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)$

$$=\int_0^1\sqrt{x}\,dx$$

$$=\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1=\frac{2}{3}$$



(1) $\lim_{n\to\infty}\frac{1}{n\sqrt{n}}(\sqrt{n+1}+\sqrt{n+2}+\cdots+\sqrt{2n})$

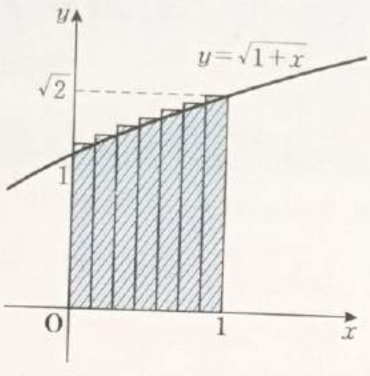
$$=\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\cdots+\sqrt{1+\frac{n}{n}}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt{1+\frac{k}{n}}$$

$$=\int_0^1\sqrt{1+x}\,dx$$

$$=\left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1$$

$$=\frac{2}{3}(2\sqrt{2}-1)$$



O123a KUMON

Integration by Quadrature and Proof of Inequalities

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Find the following limit values using definite integrals.

(1) $\displaystyle \lim_{n\to\infty}\frac{\pi}{n}\left(\sin^2\frac{\pi}{n}+\sin^2\frac{2\pi}{n}+\cdots+\sin^2\frac{n\pi}{n}\right)$

$\displaystyle =\lim_{n\to\infty}\pi\cdot\frac{1}{n}\left[\sin^2\left(\pi\cdot\frac{1}{n}\right)+\sin^2\left(\pi\cdot\frac{2}{n}\right)+\cdots+\sin^2\left(\pi\cdot\frac{n}{n}\right)\right]$

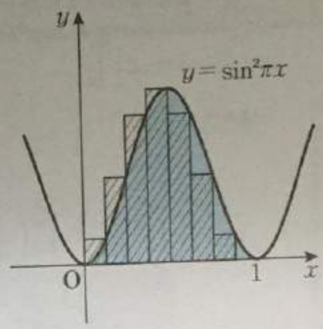
$\displaystyle =\lim_{n\to\infty}\pi\cdot\frac{1}{n}\sum_{k=1}^{n}\sin^2\left(\pi\cdot\frac{k}{n}\right)$

$\displaystyle =\pi\int_0^1\sin^2\pi x\,dx$

$\displaystyle =\frac{\pi}{2}\int_0^1(1-\cos 2\pi x)\,dx$

$\displaystyle =\frac{\pi}{2}\left[x-\frac{1}{2\pi}\sin 2\pi x\right]_0^1$

$\displaystyle =\frac{\pi}{2}$



(2) $\displaystyle \lim_{n\to\infty}\frac{1}{2n}\left(e^{\frac{1}{2n}}+e^{\frac{2}{2n}}+\cdots+e^{\frac{n}{2n}}\right)$

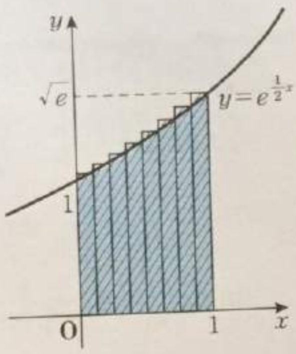
$\displaystyle =\lim_{n\to\infty}\frac{1}{2}\cdot\frac{1}{n}\left(e^{\frac{1}{2}\cdot\frac{1}{n}}+e^{\frac{1}{2}\cdot\frac{2}{n}}+\cdots+e^{\frac{1}{2}\cdot\frac{n}{n}}\right)$

$\displaystyle =\lim_{n\to\infty}\frac{1}{2}\cdot\frac{1}{n}\sum_{k=1}^{n}e^{\frac{1}{2}\cdot\frac{k}{n}}$

$\displaystyle =\frac{1}{2}\int_0^1 e^{\frac{1}{2}x}\,dx$

$\displaystyle =\frac{1}{2}\left[2e^{\frac{1}{2}x}\right]_0^1$

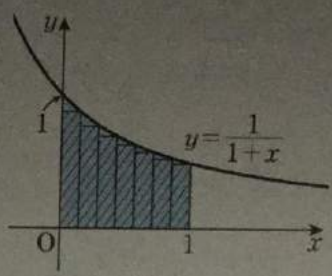
$=\sqrt{e}-1$



0123b

(3) $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)$

$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}}\right)$



$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}$

$=\int_0^1\frac{dx}{1+x}$

$=\Big[\ln|1+x|\Big]_0^1$

$=\ln 2$

(4) $\lim_{n\to\infty}\ln\left(\frac{n+1}{n}\right)^{\frac{1}{n}}\left(\frac{n+2}{n}\right)^{\frac{1}{n}}\cdots\left(\frac{2n}{n}\right)^{\frac{1}{n}}$

$=\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^{\frac{1}{n}}\left(1+\frac{2}{n}\right)^{\frac{1}{n}}\cdots\left(1+\frac{n}{n}\right)^{\frac{1}{n}}$ $\ln MN=\ln M+\ln N$

$=\lim_{n\to\infty}\left[\ln\left(1+\frac{1}{n}\right)^{\frac{1}{n}}+\ln\left(1+\frac{2}{n}\right)^{\frac{1}{n}}+\cdots+\ln\left(1+\frac{n}{n}\right)^{\frac{1}{n}}\right]$

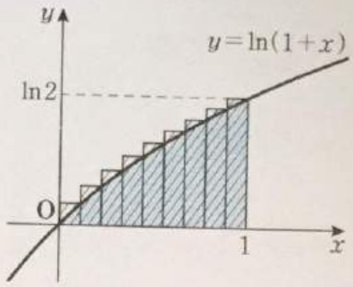
$\ln M^n=n\ln M$

$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)$

$=\int_0^1\ln(1+x)\,dx$

$=\int_0^1(x)'\ln(1+x)\,dx$

$=\Big[x\ln(1+x)\Big]_0^1-\int_0^1 x[\ln(1+x)]'dx$



$=\ln 2-\int_0^1\frac{x}{1+x}dx$

$=\ln 2-\int_0^1\left(1-\frac{1}{1+x}\right)dx$

$=\ln 2-\Big[x-\ln(1+x)\Big]_0^1$

$=2\ln 2-1$

Integration by Quadrature and Proof of Inequalities

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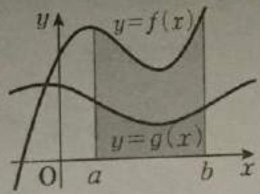
Definite Integrals and Inequalities

On the interval $[a, b]$,

if $f(x) \geqslant g(x)$, then $\int_a^b f(x)dx \geqslant \int_a^b g(x)dx$

The equality sign holds only when $f(x)=g(x)$ for all values of x.



Ex. When $0 \leqslant x \leqslant 1$, prove the inequality $\frac{1}{2} < \int_0^1 \frac{dx}{x^2+x+1} < \ln 2$ using $x+1 \leqslant x^2+x+1 \leqslant (x+1)^2$.

[Sol] Since $0 < x+1 \leqslant x^2+x+1 \leqslant (x+1)^2$ when $0 \leqslant x \leqslant 1$,

$$\frac{1}{(x+1)^2} \leqslant \frac{1}{x^2+x+1} \leqslant \frac{1}{x+1}$$

When $0 < a \leqslant b \leqslant c$, $\frac{1}{a} \geqslant \frac{1}{b} \geqslant \frac{1}{c}$

Also, the equality sign does not hold for all values of x.

$$\therefore \int_0^1 \frac{dx}{(x+1)^2} < \int_0^1 \frac{dx}{x^2+x+1} < \int_0^1 \frac{dx}{x+1}$$

$$\therefore \left[-\frac{1}{x+1}\right]_0^1 < \int_0^1 \frac{dx}{x^2+x+1} < \left[\ln|x+1|\right]_0^1$$

$$\therefore \frac{1}{2} < \int_0^1 \frac{dx}{x^2+x+1} < \ln 2$$

1. When $0 \leqslant x \leqslant 1$, prove the inequality $\ln 2 < \int_0^1 \frac{dx}{1+x^2} < 1$ using $1 \leqslant 1+x^2 \leqslant 1+x$.

[Sol] Since $1 \leqslant 1+x^2 \leqslant 1+x$ when $0 \leqslant x \leqslant 1$, $\frac{1}{1+x} \leqslant \frac{1}{1+x^2} \leqslant 1$

Also, the equality sign does not hold for all values of x.

$$\therefore \int_0^1 \frac{dx}{1+x} < \int_0^1 \frac{dx}{1+x^2} < \int_0^1 dx$$

$$\therefore \left[\ln|1+x|\right]_0^1 < \int_0^1 \frac{dx}{1+x^2} < \left[x\right]_0^1$$

$$\therefore \ln 2 < \int_0^1 \frac{dx}{1+x^2} < 1$$

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O124b

2. When $0 \leqslant x \leqslant \frac{\pi}{4}$, prove the inequality $\frac{\pi}{4} < \int_0^{\frac{\pi}{4}} \frac{dx}{\sqrt{1-\sin x}} < 2-\sqrt{4-\pi}$ using $0 \leqslant \sin x \leqslant x$.

[Sol] Since $0 \leqslant \sin x \leqslant x$ when $0 \leqslant x \leqslant \frac{\pi}{4}$, $0 < 1-x \leqslant 1-\sin x \leqslant 1$; therefore,

※

$$\sqrt{1-x} \leqslant \sqrt{1-\sin x} \leqslant 1, \text{ i.e. } 1 \leqslant \frac{1}{\sqrt{1-\sin x}} \leqslant \frac{1}{\sqrt{1-x}}$$

Also, the equality sign does not hold for all values of x.

$$\therefore \int_0^{\frac{\pi}{4}} dx < \int_0^{\frac{\pi}{4}} \frac{dx}{\sqrt{1-\sin x}} < \int_0^{\frac{\pi}{4}} \frac{dx}{\sqrt{1-x}}$$

$$\therefore \Big[x\Big]_0^{\frac{\pi}{4}} < \int_0^{\frac{\pi}{4}} \frac{dx}{\sqrt{1-\sin x}} < \Big[-2(1-x)^{\frac{1}{2}}\Big]_0^{\frac{\pi}{4}}$$

$$\therefore \frac{\pi}{4} < \int_0^{\frac{\pi}{4}} \frac{dx}{\sqrt{1-\sin x}} < 2-\sqrt{4-\pi}$$

※ Multiplying each side of $0 \leqslant \sin x \leqslant x$ by -1, then adding 1, $1-x \leqslant 1-\sin x \leqslant 1$

$-2\sqrt{1-\frac{\pi}{4}}+2$
$=2-2\sqrt{\frac{4-\pi}{4}}$

3. When $0 \leqslant x \leqslant 1$, prove the inequality $2\left(1-\frac{1}{\sqrt{e}}\right) < \int_0^1 e^{-\frac{1}{2}x^2} dx < 1$ using $x^2 \leqslant x$.

[Sol] Since $0 \leqslant x^2 \leqslant x$ when $0 \leqslant x \leqslant 1$, $-\frac{1}{2}x \leqslant -\frac{1}{2}x^2 \leqslant 0$

Since $e > 1$, $e^{-\frac{1}{2}x} \leqslant e^{-\frac{1}{2}x^2} \leqslant e^0$, i.e. $e^{-\frac{1}{2}x} \leqslant e^{-\frac{1}{2}x^2} \leqslant 1$

Also, the equality sign does not hold for all values of x.

$$\therefore \int_0^1 e^{-\frac{1}{2}x} dx < \int_0^1 e^{-\frac{1}{2}x^2} dx < \int_0^1 dx$$

$$\therefore \Big[-2e^{-\frac{1}{2}x}\Big]_0^1 < \int_0^1 e^{-\frac{1}{2}x^2} dx < \Big[x\Big]_0^1$$

$$\therefore 2\left(1-\frac{1}{\sqrt{e}}\right) < \int_0^1 e^{-\frac{1}{2}x^2} dx < 1$$

When $a > 1$, $p \leqslant q \Leftrightarrow a^p \leqslant a^q$ (K187)

Integration by Quadrature and Proof of Inequalities

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1. When $0 \leqslant x \leqslant 1$, prove the inequality $\frac{\pi}{4} < \int_0^1 \frac{dx}{x^3+1} < 1$ using $1 \leqslant x^3+1 \leqslant x^2+1$.

[Sol] Since $1 \leqslant x^3+1 \leqslant x^2+1$ when $0 \leqslant x \leqslant 1$, $\frac{1}{x^2+1} \leqslant \frac{1}{x^3+1} \leqslant 1$

Also, the equality sign does not hold for all values of x.

$$\therefore \int_0^1 \frac{dx}{x^2+1} < \int_0^1 \frac{dx}{x^3+1} < \int_0^1 dx$$

Then, let $\int_0^1 \frac{dx}{x^2+1} = I$ and $x = \tan\theta$. $dx = \frac{d\theta}{\cos^2\theta}$ ← O106

$$x^2+1 = \tan^2\theta + 1 = \frac{1}{\cos^2\theta}$$

$$\begin{array}{c|c} x & 0 \longrightarrow 1 \\ \hline \theta & 0 \longrightarrow \frac{\pi}{4} \end{array}$$

$$\therefore I = \int_0^{\frac{\pi}{4}} \frac{1}{\frac{1}{\cos^2\theta}} \cdot \frac{d\theta}{\cos^2\theta}$$

$$= \int_0^{\frac{\pi}{4}} d\theta$$

$$= \Big[\theta\Big]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4}$$

Also, $\int_0^1 dx = \Big[x\Big]_0^1 = 1$

$$\therefore \frac{\pi}{4} < \int_0^1 \frac{dx}{x^3+1} < 1$$

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O125b

2. Solve the following questions.

(1) Find the definite integral $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$.

[Sol] Let $x=\sin\theta$. $dx=\cos\theta\, d\theta$ ⟵ O105

x	$0 \longrightarrow \frac{1}{2}$
θ	$0 \longrightarrow \frac{\pi}{6}$

Since $\cos\theta > 0$ when $0 \leqslant \theta \leqslant \frac{\pi}{6}$,

$$\sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\cos\theta$$

$$\therefore \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \int_0^{\frac{\pi}{6}} \frac{1}{\cos\theta}\cdot\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} d\theta$$

$$= \Big[\theta\Big]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{6}$$

(2) Prove the inequality $\frac{1}{2} < \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^3}} < \frac{\pi}{6}$.

[Sol] Since $0 \leqslant x^3 \leqslant x^2$ when $0 \leqslant x \leqslant \frac{1}{2}$, $0 < 1-x^2 \leqslant 1-x^3 \leqslant 1$; therefore,

$\sqrt{1-x^2} \leqslant \sqrt{1-x^3} \leqslant 1$, i.e. $1 \leqslant \frac{1}{\sqrt{1-x^3}} \leqslant \frac{1}{\sqrt{1-x^2}}$ ⟵ ※

Also, the equality sign does not hold for all values of x.

$$\therefore \int_0^{\frac{1}{2}} dx < \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^3}} < \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$$

$$\int_0^{\frac{1}{2}} dx = \Big[x\Big]_0^{\frac{1}{2}} = \frac{1}{2}$$

From (1), $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{6}$

$$\therefore \frac{1}{2} < \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^3}} < \frac{\pi}{6}$$

※ For the inequality to be proved, find $f(x)$ and $g(x)$ such that $f(x) \leqslant \frac{1}{\sqrt{1-x^3}} \leqslant g(x)$ and also $\int_0^{\frac{1}{2}} f(x)dx = \frac{1}{2}$, $\int_0^{\frac{1}{2}} g(x)dx = \frac{\pi}{6}$.

Integration by Quadrature and Proof of Inequalities

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Ex. Given that n is a natural number, prove the following inequality.

$$\ln(n+1) < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

[Sol] Let k be a natural number. When $k \leqslant x \leqslant k+1$, $\frac{1}{x} \leqslant \frac{1}{k}$

Also, the equality sign does not hold for all values of x.

$$\therefore \int_k^{k+1}\frac{dx}{x} < \int_k^{k+1}\frac{dx}{k}$$

When $0 < a \leqslant b$, $\frac{1}{a} \geqslant \frac{1}{b}$

$$\therefore \int_k^{k+1}\frac{dx}{x} < \boxed{\frac{1}{k}} \quad \cdots ①$$

$\int_k^{k+1}\frac{dx}{k} = \frac{1}{k}\Big[x\Big]_k^{k+1}$

Substituting $k=1, 2, 3, \ldots, n$ into ① and adding up the terms on each side,

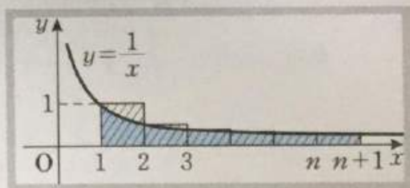
$$\sum_{k=1}^{n}\int_k^{k+1}\frac{dx}{x} < \sum_{k=1}^{n}\boxed{\frac{1}{k}}$$

$$\text{LHS} = \int_1^2\frac{dx}{x}+\int_2^3\frac{dx}{x}+\int_3^4\frac{dx}{x}+\cdots+\int_n^{\boxed{n+1}}\frac{dx}{x}$$

$$= \int_1^{\boxed{n+1}}\frac{dx}{x}$$

$$= \Big[\ln|x|\Big]_1^{\boxed{n+1}}$$

$$= \ln(\boxed{n+1})$$



The proved inequality shows the comparison of the areas in the diagram.

$$\text{RHS} = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

$$\therefore \ln(n+1) < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

Answers: $\frac{1}{k}$, $\frac{1}{k}$, $n+1$, $n+1$, $n+1$, $n+1$

Consider the limits of both sides of the inequality shown in **Ex.**.

Since $\lim_{n\to\infty}\ln(n+1)=\infty$, and from Limits of Sequences and Their Relationships (N67),

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)=\infty$$

O126b

1. Given that n is a natural number, prove the following inequality.

$$2(\sqrt{n+1}-1)<1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}$$

[Sol] Let k be a natural number. When $k \leqslant x \leqslant k+1$, $\frac{1}{\sqrt{x}} \leqslant \frac{1}{\sqrt{k}}$

Also, the equality sign does not hold for all values of x.

$$\therefore \int_k^{k+1}\frac{dx}{\sqrt{x}}<\int_k^{k+1}\frac{dx}{\sqrt{k}}$$

$$\therefore \int_k^{k+1}\frac{dx}{\sqrt{x}}<\frac{1}{\sqrt{k}} \quad \cdots①$$

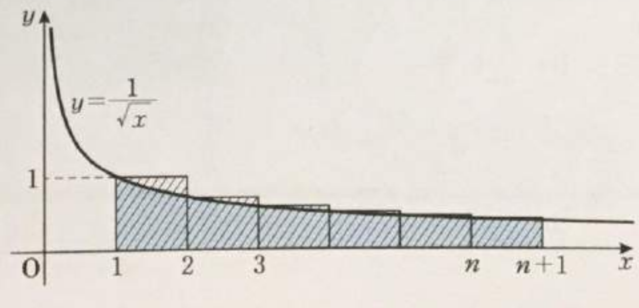
Substituting $k=1, 2, 3, \ldots, n$ into ① and adding up the terms on each side,

$$\sum_{k=1}^{n}\int_k^{k+1}\frac{dx}{\sqrt{x}}<\sum_{k=1}^{n}\frac{1}{\sqrt{k}}$$

$$\begin{aligned}\text{LHS} &= \int_1^2\frac{dx}{\sqrt{x}}+\int_2^3\frac{dx}{\sqrt{x}}+\int_3^4\frac{dx}{\sqrt{x}}+\cdots+\int_n^{n+1}\frac{dx}{\sqrt{x}}\\ &=\int_1^{n+1}\frac{dx}{\sqrt{x}}\\ &=\left[2x^{\frac{1}{2}}\right]_1^{n+1}\\ &=2(\sqrt{n+1}-1)\end{aligned}$$

$$\text{RHS} =1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}$$

$$\therefore 2(\sqrt{n+1}-1)<1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}$$



Integration by Quadrature and Proof of Inequalities

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1. Given that n is a natural number greater than or equal to 2, prove the following inequality.

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}<\ln n$$

[Sol] Let k be a natural number. When $k \leqslant x \leqslant k+1$, $\frac{1}{k+1} \leqslant \frac{1}{x}$

Also, the equality sign does not hold for all values of x.

$$\therefore \int_k^{k+1}\frac{dx}{k+1}<\int_k^{k+1}\frac{dx}{x}$$

$$\therefore \frac{1}{k+1}<\int_k^{k+1}\frac{dx}{x} \quad \cdots①$$

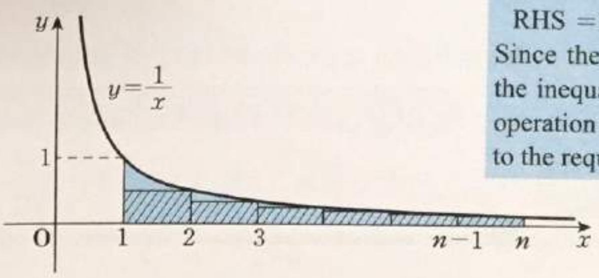
Substituting $k=1,\ 2,\ 3,\ \ldots,\ n-1$ into ① and adding up the terms on each side, when $n \geqslant 2$,

$$\sum_{k=1}^{n-1}\frac{1}{k+1}<\sum_{k=1}^{n-1}\int_k^{k+1}\frac{dx}{x}$$

$$\text{LHS} =\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}$$

$$\begin{aligned}\text{RHS} &= \int_1^2\frac{dx}{x}+\int_2^3\frac{dx}{x}+\int_3^4\frac{dx}{x}+\cdots+\int_{n-1}^{n}\frac{dx}{x}\\ &=\int_1^n\frac{dx}{x}\\ &=\Big[\ln|x|\Big]_1^n\\ &=\ln n\end{aligned}$$

$$\therefore \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}<\ln n$$



Substituting $k=1,\ 2,\ 3,\ \ldots,\ n-1,\ n$ into ① and adding up the terms on each side,

$$\text{LHS} =\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\frac{1}{n+1}$$

$$\text{RHS} =\ln(n+1)$$

Since the LHS and the RHS cannot form the inequality to be proved, consider what operation has to be done to make them lead to the required inequality.

0127b

2. Given that n is a natural number greater than or equal to 2, prove the following inequality.

$$1-\frac{1}{n}+\frac{1}{n^2}<1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}$$

[Sol] Let k be a natural number. When $k \leqslant x \leqslant k+1$, $\frac{1}{x^2} \leqslant \frac{1}{k^2}$

Also, the equality sign does not hold for all values of x.

$$\therefore \int_k^{k+1} \frac{dx}{x^2} < \int_k^{k+1} \frac{dx}{k^2}$$

$$\therefore \int_k^{k+1} \frac{dx}{x^2} < \frac{1}{k^2} \quad \cdots ①$$

Substituting $k=1, 2, 3, \ldots, n-1$ into ① and adding up the terms on each side, when $n \geqslant 2$,

$$\sum_{k=1}^{n-1} \int_k^{k+1} \frac{dx}{x^2} < \sum_{k=1}^{n-1} \frac{1}{k^2}$$

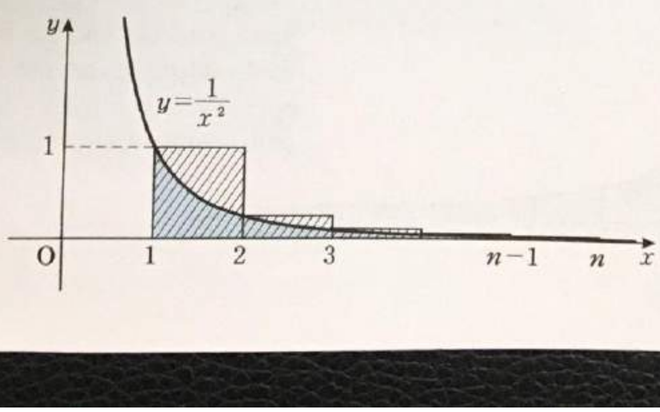
$$\begin{aligned} \text{LHS } &= \int_1^2 \frac{dx}{x^2} + \int_2^3 \frac{dx}{x^2} + \int_3^4 \frac{dx}{x^2} + \cdots + \int_{n-1}^{n} \frac{dx}{x^2} \\ &= \int_1^n \frac{dx}{x^2} \\ &= \left[-\frac{1}{x}\right]_1^n \\ &= 1-\frac{1}{n} \end{aligned}$$

$$\text{RHS } = 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(n-1)^2}$$

$$\therefore 1-\frac{1}{n} < 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(n-1)^2}$$

Adding $\frac{1}{n^2}$ to both sides,

$$1-\frac{1}{n}+\frac{1}{n^2}<1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}$$



Integration by Quadrature and Proof of Inequalities

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Let $a < b$. Using the properties that inequality $\int_a^b [tf(x)+g(x)]^2 dx \geqq 0$ is true for any arbitrary real number t and that the value of the definite integral of this inequality is equal to a quadratic expression in t, prove the following inequality.

$$\left[\int_a^b f(x)g(x)dx\right]^2 \leqq \left\{\int_a^b [f(x)]^2 dx\right\}\left\{\int_a^b [g(x)]^2 dx\right\} \cdots Ⓐ$$

[Sol] Let $\int_a^b [f(x)]^2 dx = A$, $\int_a^b f(x)g(x)dx = B$ and $\int_a^b [g(x)]^2 dx = C$.

(i) In the case where $f(x)=0$ or $g(x)=0$ for all values of x, both sides of inequality Ⓐ become 0, therefore Ⓐ is true.

(ii) In cases other than (i),

Since $[f(x)]^2 \geqq 0$, $A = \int_a^b [f(x)]^2 dx > 0$

$\int_a^b [f(x)]^2 dx = 0$ is true only when $f(x)=0$ for all values of x.

$$\int_a^b [tf(x)+g(x)]^2 dx$$
$$= t^2\int_a^b [f(x)]^2 dx + 2t\int_a^b f(x)g(x)dx + \int_a^b [g(x)]^2 dx$$
$$= At^2 + 2Bt + C$$

Since $\int_a^b [tf(x)+g(x)]^2 dx \geqq 0$, inequality $At^2+2Bt+C \geqq 0$ is true for any arbitrary real number t. Therefore, let D be the discriminant of $At^2+2Bt+C=0$. Then, $D \leqq \boxed{0}$.

$$\therefore \frac{D}{4} = B^2 - AC \leqq \boxed{0} \quad \therefore B^2 \leqq AC$$

Therefore, Ⓐ is true.

From (i) and (ii), Ⓐ holds true.

Answers: All the answers are the same, 0

The inequality Ⓐ proved above is called the ***Cauchy–Schwarz Inequality***.

Cauchy–Schwarz Inequality

$$\left[\int_a^b f(x)g(x)dx\right]^2 \leqq \left\{\int_a^b [f(x)]^2 dx\right\}\left\{\int_a^b [g(x)]^2 dx\right\}$$

0128b

Prove the following inequalities using the Cauchy–Schwarz Inequality.

Ex. $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{x\sin x}\,dx \leqslant \frac{\sqrt{2}}{4}\pi$

[Sol] From the Cauchy–Schwarz Inequality,

$$\left(\int_0^{\frac{\pi}{2}} \sqrt{x\sin x}\,dx\right)^2 \leqslant \left(\int_0^{\frac{\pi}{2}} x\,dx\right)\left(\int_0^{\frac{\pi}{2}} \sin x\,dx\right)$$

In the Cauchy–Schwarz Inequality, $f(x)=\sqrt{x}$, $g(x)=\sqrt{\sin x}$, $a=0$, $b=\frac{\pi}{2}$

Then, $\displaystyle\int_0^{\frac{\pi}{2}} x\,dx=\left[\frac{1}{2}x^2\right]_0^{\frac{\pi}{2}}=\frac{\pi^2}{8}$

$$\int_0^{\frac{\pi}{2}} \sin x\,dx=\Big[-\cos x\Big]_0^{\frac{\pi}{2}}=1$$

$$\therefore \left(\int_0^{\frac{\pi}{2}} \sqrt{x\sin x}\,dx\right)^2 \leqslant \frac{\pi^2}{8}$$

Since $\sqrt{x\sin x} \geqslant 0$ when $0 \leqslant x \leqslant \frac{\pi}{2}$,

$$\int_0^{\frac{\pi}{2}} \sqrt{x\sin x}\,dx \geqslant 0$$

If $f(x) \geqslant 0$ on the interval $[a, b]$, $\displaystyle\int_a^b f(x)dx \geqslant 0$

$$\therefore \int_0^{\frac{\pi}{2}} \sqrt{x\sin x}\,dx \leqslant \frac{\sqrt{2}}{4}\pi$$

When $A \geqslant 0$ and $B \geqslant 0$, if $A^2 \leqslant B^2$, then $A \leqslant B$

(1) $\displaystyle\int_1^e \sqrt{\ln x}\,dx \leqslant \sqrt{e-1}$

[Sol] From the Cauchy–Schwarz Inequality,

$$\left(\int_1^e \sqrt{\ln x}\,dx\right)^2 \leqslant \left(\int_1^e dx\right)\left(\int_1^e \ln x\,dx\right)$$

In the Cauchy–Schwarz Inequality, $f(x)=1$, $g(x)=\sqrt{\ln x}$, $a=1$, $b=e$

Then, $\displaystyle\int_1^e dx=\Big[x\Big]_1^e=e-1$

$$\begin{aligned}\int_1^e \ln x\,dx &= \int_1^e (x)'\ln x\,dx\\ &=\Big[x\ln x\Big]_1^e-\int_1^e x(\ln x)'dx\\ &=e-\int_1^e dx\\ &=e-\Big[x\Big]_1^e\\ &=1\end{aligned}$$

$$\therefore \left(\int_1^e \sqrt{\ln x}\,dx\right)^2 \leqslant e-1$$

Since $\sqrt{\ln x} \geqslant 0$ when $1 \leqslant x \leqslant e$,

$$\int_1^e \sqrt{\ln x}\,dx \geqslant 0$$

$$\therefore \int_1^e \sqrt{\ln x}\,dx \leqslant \sqrt{e-1}$$

Integration by Quadrature and Proof of Inequalities

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1. Given that n is a positive integer and P_k is the point on curve $y=\ln x$ with the x-coordinate $1+\frac{k}{n}$. (k is a positive integer less than or equal to n.)

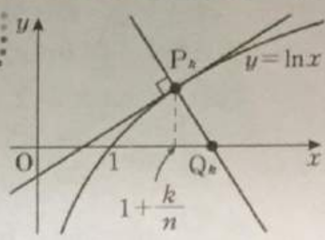
(1) Let Q_k be the point of intersection of the normal to $y=\ln x$ at point P_k with the x-axis. Find the coordinates of point Q_k.

The equation of the normal at point $(a, f(a))$ is, when $f'(a)\neq 0$, $y-f(a)=-\frac{1}{f'(a)}(x-a)$

[Sol] Since $y=\ln x$, $y'=\frac{1}{x}$

Therefore, the equation of the normal at P_k is

$$y-\ln\left(1+\frac{k}{n}\right)=-\left(1+\frac{k}{n}\right)\left[x-\left(1+\frac{k}{n}\right)\right]$$



When $y=0$, $x=1+\frac{k}{n}+\frac{1}{1+\frac{k}{n}}\ln\left(1+\frac{k}{n}\right)$

$$\therefore\ \mathrm{Q}_k\left(1+\frac{k}{n}+\frac{1}{1+\frac{k}{n}}\ln\left(1+\frac{k}{n}\right),\ 0\right)\ \left[\mathrm{Q}_k\left(1+\frac{k}{n}+\frac{n}{n+k}\ln\left(1+\frac{k}{n}\right),\ 0\right)\right]$$

(2) Let l_k be the length of the line segment connecting the origin and point Q_k. Find $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n} l_k$.

[Sol] From (1), since $l_k=1+\frac{k}{n}+\frac{1}{1+\frac{k}{n}}\ln\left(1+\frac{k}{n}\right)$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}l_k=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[1+\frac{k}{n}+\frac{1}{1+\frac{k}{n}}\ln\left(1+\frac{k}{n}\right)\right]$$

$$=\int_0^1\left[1+x+\frac{1}{1+x}\ln(1+x)\right]dx$$

Since $\int_0^1(1+x)dx=\left[x+\frac{1}{2}x^2\right]_0^1=\frac{3}{2}$, find $\int_0^1\frac{1}{1+x}\ln(1+x)dx$.

Let $\ln(1+x)=t$. $\frac{1}{1+x}dx=dt$

x	$0\to 1$
t	$0\to \ln 2$

$$\therefore\ \int_0^1\frac{1}{1+x}\ln(1+x)dx=\int_0^{\ln 2}t\,dt=\left[\frac{1}{2}t^2\right]_0^{\ln 2}=\frac{1}{2}(\ln 2)^2$$

$$\therefore\ \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}l_k=\frac{3}{2}+\frac{1}{2}(\ln 2)^2$$

0129b

2. Given that n is a natural number greater than or equal to 2, prove the following inequality.

$$n\ln n-n+1<\ln(n!)<(n+1)\ln n-n+1$$

$$(n!=n(n-1)(n-2)\cdot\cdots\cdot 3\cdot 2\cdot 1)$$

[Sol] Let k be a natural number. When $k\leqslant x\leqslant k+1$, $\ln k\leqslant \ln x\leqslant \ln(k+1)$

Also, the equality sign does not hold for all values of x.

$$\therefore \int_k^{k+1}\ln k\,dx<\int_k^{k+1}\ln x\,dx<\int_k^{k+1}\ln(k+1)dx$$

$$\therefore \ln k<\int_k^{k+1}\ln x\,dx<\ln(k+1)\ \cdots①$$

Substituting $k=1, 2, 3, \ldots, n-1$ into ① and adding up the terms on each side, when $n\geqslant 2$,

$$\sum_{k=1}^{n-1}\ln k<\sum_{k=1}^{n-1}\int_k^{k+1}\ln x\,dx<\sum_{k=1}^{n-1}\ln(k+1)\ \cdots②$$

$$\begin{aligned}\sum_{k=1}^{n-1}\ln k&=\ln 1+\ln 2+\ln 3+\cdots+\ln(n-1)\\&=\ln[1\cdot 2\cdot 3\cdot\cdots\cdot(n-1)]\\&=\ln[(n-1)!]\ \cdots③\end{aligned}$$

$\ln M+\ln N$
$=\ln MN$

$$\begin{aligned}\sum_{k=1}^{n-1}\int_k^{k+1}\ln x\,dx&=\int_1^2\ln x\,dx+\int_2^3\ln x\,dx+\int_3^4\ln x\,dx+\cdots+\int_{n-1}^n\ln x\,dx\\&=\int_1^n\ln x\,dx\\&=\int_1^n(x)'\ln x\,dx\\&=\Big[x\ln x\Big]_1^n-\int_1^n x(\ln x)'dx\\&=n\ln n-\int_1^n dx\\&=n\ln n-\Big[x\Big]_1^n\\&=n\ln n-n+1\ \cdots④\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{n-1}\ln(k+1)&=\ln 2+\ln 3+\ln 4+\cdots+\ln n\\&=\ln(2\cdot 3\cdot 4\cdot\cdots\cdot n)\\&=\ln(n!)\ \cdots⑤\end{aligned}$$

$\ln(2\cdot 3\cdot 4\cdot\cdots\cdot n)$
$=\ln(1\cdot 2\cdot 3\cdot 4\cdot\cdots\cdot n)$
$=\ln(n!)$

From ②~④, $\ln[(n-1)!]<n\ln n-n+1$

Adding $\ln n$ to both sides, $\ln(n!)<(n+1)\ln n-n+1\ \cdots⑥$

Also, from ②, ④ and ⑤, $n\ln n-n+1<\ln(n!)\ \cdots⑦$

From ⑥ and ⑦, $n\ln n-n+1<\ln(n!)<(n+1)\ln n-n+1$

Integration by Quadrature and Proof of Inequalities

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1. Find the limit value using a definite integral. ➡ O122

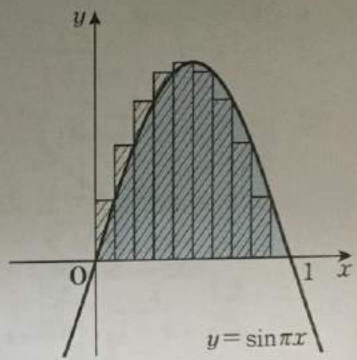
$$\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\frac{n\pi}{n}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sin\left(\pi\cdot\frac{k}{n}\right)$$

$$=\int_0^1 \sin\pi x\,dx$$

$$=\left[-\frac{1}{\pi}\cos\pi x\right]_0^1$$

$$=\frac{2}{\pi}$$



2. When $0\leqq x\leqq 2$, prove the inequality $\frac{2}{3}<\int_0^2\frac{dx}{x^3+1}<2$ using $1\leqq x^3+1\leqq(x+1)^2$. ➡ O124

[Sol] Since $1\leqq x^3+1\leqq(x+1)^2$ when $0\leqq x\leqq 2$, $\frac{1}{(x+1)^2}\leqq\frac{1}{x^3+1}\leqq 1$

Also, the equality sign does not hold for all values of x.

$$\therefore\ \int_0^2\frac{dx}{(x+1)^2}<\int_0^2\frac{dx}{x^3+1}<\int_0^2 dx$$

$$\therefore\ \left[-\frac{1}{x+1}\right]_0^2<\int_0^2\frac{dx}{x^3+1}<\left[x\right]_0^2$$

$$\therefore\ \frac{2}{3}<\int_0^2\frac{dx}{x^3+1}<2$$

O130b

3. Given that n is a natural number greater than or equal to 2, prove the following inequality.

➡ O126

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}<2\sqrt{n}-1$$

[Sol] Let k be a natural number. When $k \leqslant x \leqslant k+1$, $\frac{1}{\sqrt{k+1}} \leqslant \frac{1}{\sqrt{x}}$

Also, the equality sign does not hold for all values of x.

$$\therefore \int_k^{k+1} \frac{dx}{\sqrt{k+1}} < \int_k^{k+1} \frac{dx}{\sqrt{x}}$$

$$\therefore \frac{1}{\sqrt{k+1}} < \int_k^{k+1} \frac{dx}{\sqrt{x}} \quad \cdots ①$$

Substituting $k=1, 2, 3, \ldots, n-1$ into ① and adding up the terms on each side, when $n \geqslant 2$,

$$\sum_{k=1}^{n-1} \frac{1}{\sqrt{k+1}} < \sum_{k=1}^{n-1} \int_k^{k+1} \frac{dx}{\sqrt{x}}$$

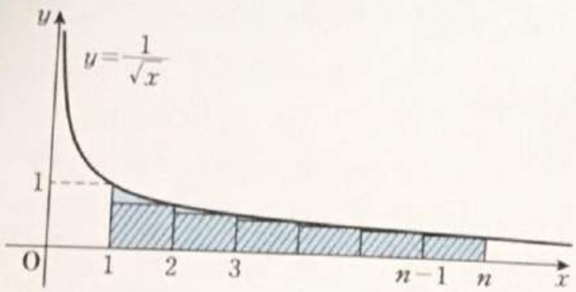
$$\text{LHS} = \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{n}}$$

$$\begin{aligned}\text{RHS} &= \int_1^2 \frac{dx}{\sqrt{x}}+\int_2^3 \frac{dx}{\sqrt{x}}+\int_3^4 \frac{dx}{\sqrt{x}}+\cdots+\int_{n-1}^n \frac{dx}{\sqrt{x}}\\ &= \int_1^n \frac{dx}{\sqrt{x}}\\ &= \Big[2x^{\frac{1}{2}}\Big]_1^n\\ &= 2\sqrt{n}-2\end{aligned}$$

$$\therefore \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{n}} < 2\sqrt{n}-2$$

Adding 1 to both sides,

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}<2\sqrt{n}-1$$



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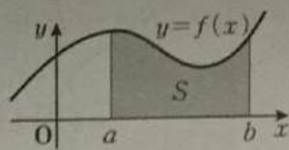
Given area S enclosed by the curve $y=f(x)$, the x-axis and two lines $x=a$ and $x=b$ $(a<b)$, the following formulas are true.

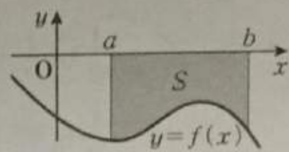


Curve $y=f(x)$ and Area

On the interval $[a, b]$,

when $f(x) \geqq 0$, $S=\int_a^b f(x)dx$

when $f(x) \leqq 0$, $S=-\int_a^b f(x)dx$



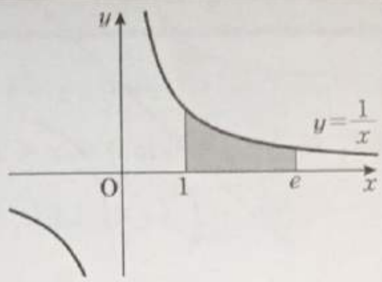
Find the area S enclosed by the following curves, lines and the x-axis.

Ex. $y=\dfrac{1}{x}$, $x=1$, $x=e$

[Sol] Since $y>0$ in $1 \leqq x \leqq e$,

$$S=\int_1^e \frac{dx}{x}$$

$$=\Big[\ln|x|\Big]_1^e=1$$

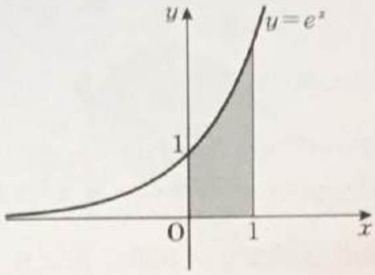


(1) $y=e^x$, $x=0$, $x=1$

[Sol] Since $y>0$ in $0 \leqq x \leqq 1$,

$$S=\int_0^1 e^x dx$$

$$=\Big[e^x\Big]_0^1=e-1$$

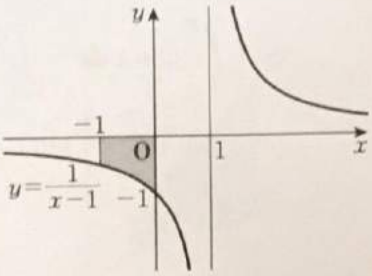


(2) $y=\dfrac{1}{x-1}$, $x=-1$, $x=0$

[Sol] Since $y<0$ in $-1 \leqq x \leqq 0$,

$$S=-\int_{-1}^0 \frac{dx}{x-1}$$

$$=-\Big[\ln|x-1|\Big]_{-1}^0=\ln 2$$



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0131b

Ex. $y=\ln x,\ \ x=\dfrac{1}{2},\ \ x=e$

[Sol] Since $y \leqslant 0$ in $\dfrac{1}{2} \leqslant x \leqslant 1$ and $y \geqslant 0$ in $1 \leqslant x \leqslant e$,

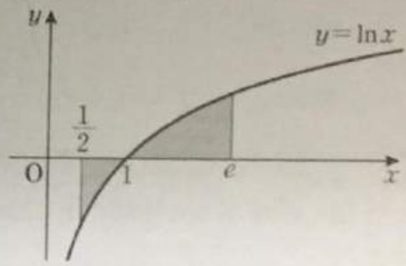
$$S=-\int_{\frac{1}{2}}^{1}\ln x\,dx+\int_{1}^{e}\ln x\,dx$$

$$\int \ln x\,dx=\int (x)'\ln x\,dx$$
$$=x\ln x-\int x(\ln x)'dx=x\ln x-\int x\cdot\frac{1}{x}\,dx$$

$$=-\left(\Big[x\ln x\Big]_{\frac{1}{2}}^{1}-\int_{\frac{1}{2}}^{1}dx\right)+\left(\Big[x\ln x\Big]_{1}^{e}-\int_{1}^{e}dx\right)$$

$$=\frac{1}{2}\ln\frac{1}{2}+\Big[x\Big]_{\frac{1}{2}}^{1}+e-\Big[x\Big]_{1}^{e}$$

$$=\frac{1}{2}\ln\frac{1}{2}+\frac{3}{2}$$



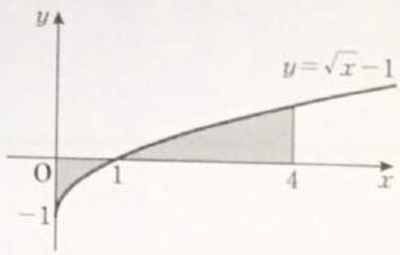
(3) $y=\sqrt{x}-1,\ \ x=0,\ \ x=4$

[Sol] Since $y \leqslant 0$ in $0 \leqslant x \leqslant 1$ and $y \geqslant 0$ in $1 \leqslant x \leqslant 4$,

$$S=-\int_{0}^{1}(\sqrt{x}-1)dx+\int_{1}^{4}(\sqrt{x}-1)dx$$

$$=-\left[\frac{2}{3}x^{\frac{3}{2}}-x\right]_{0}^{1}+\left[\frac{2}{3}x^{\frac{3}{2}}-x\right]_{1}^{4}$$

$$=2$$



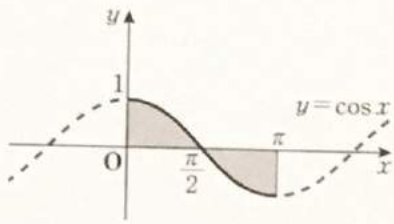
(4) $y=\cos x\ (0 \leqslant x \leqslant \pi),\ \ x=0,\ \ x=\pi$

[Sol] Since $y \geqslant 0$ in $0 \leqslant x \leqslant \dfrac{\pi}{2}$ and $y \leqslant 0$ in $\dfrac{\pi}{2} \leqslant x \leqslant \pi$,

$$S=\int_{0}^{\frac{\pi}{2}}\cos x\,dx-\int_{\frac{\pi}{2}}^{\pi}\cos x\,dx$$

$$=\Big[\sin x\Big]_{0}^{\frac{\pi}{2}}-\Big[\sin x\Big]_{\frac{\pi}{2}}^{\pi}$$

$$=2$$



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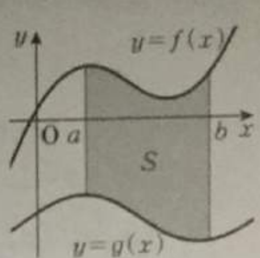
Given area S enclosed by two curves $y=f(x)$ and $y=g(x)$ and two lines $x=a$ and $x=b$ $(a<b)$, the following formula is true.



Area between Two Curves

On the interval $[a, b]$, when $f(x) \geqq g(x)$,

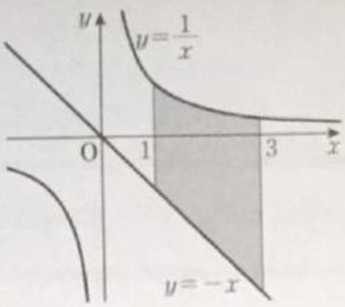
$$S=\int_a^b [f(x)-g(x)]\,dx$$

1. Find the area S enclosed by the following curves and lines.

Ex. $f(x)=\dfrac{1}{x}$, $g(x)=-x$, $x=1$, $x=3$

[Sol] Since $f(x)>g(x)$ in $1 \leqq x \leqq 3$,

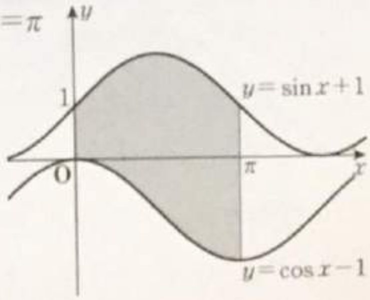
$$S=\int_1^3\left[\frac{1}{x}-(-x)\right]dx$$
$$=\int_1^3\left(\frac{1}{x}+x\right)dx$$
$$=\left[\ln|x|+\frac{1}{2}x^2\right]_1^3=\ln 3+4$$



(1) $f(x)=\sin x+1$, $g(x)=\cos x-1$, $x=0$, $x=\pi$

[Sol] Since $f(x)>g(x)$ in $0 \leqq x \leqq \pi$,

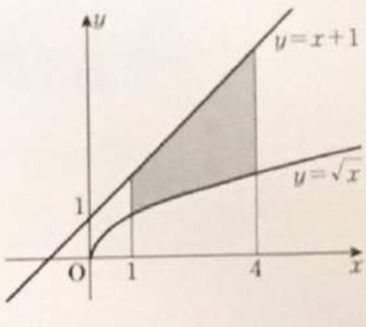
$$S=\int_0^\pi [(\sin x+1)-(\cos x-1)]\,dx$$
$$=\int_0^\pi (\sin x-\cos x+2)\,dx$$
$$=\Big[-\cos x-\sin x+2x\Big]_0^\pi=2\pi+2$$



(2) $f(x)=\sqrt{x}$, $g(x)=x+1$, $x=1$, $x=4$

[Sol] Since $f(x)<g(x)$ in $1 \leqq x \leqq 4$,

$$S=\int_1^4 [(x+1)-\sqrt{x}\,]\,dx$$
$$=\left[\frac{1}{2}x^2+x-\frac{2}{3}x^{\frac{3}{2}}\right]_1^4$$
$$=\frac{35}{6}$$



01 32b

Ex. Find the area S enclosed by the following curve and line.

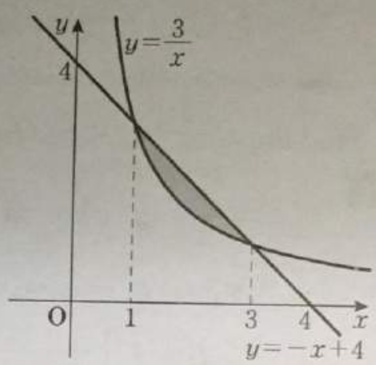
$$f(x)=-x+4,\quad g(x)=\frac{3}{x}$$

[Sol] Since $-x+4=\dfrac{3}{x}$, the x-coordinates of the points of intersection are

$$x^2-4x+3=0$$
$$(x-1)(x-3)=0$$
$$\therefore\ x=1,\ 3$$

Since $f(x)\geqslant g(x)$ in $1\leqslant x\leqslant 3$,

$$S=\int_1^3\left[(-x+4)-\frac{3}{x}\right]dx$$
$$=\left[-\frac{1}{2}x^2+4x-3\ln|x|\right]_1^3$$
$$=4-3\ln 3$$



2. Find the area S enclosed by the following two curves.

$$f(x)=\frac{1}{8}x^2,\quad g(x)=\sqrt{x}$$

[Sol] Since $\dfrac{1}{8}x^2=\sqrt{x}$, the x-coordinates of the points of intersection are

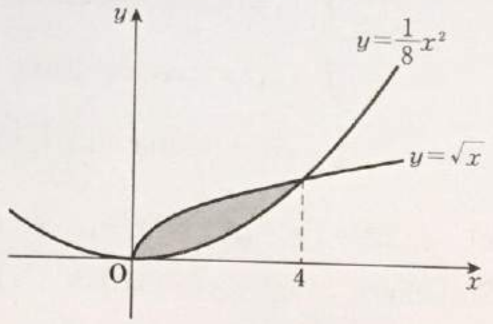
$$x^4-64x=0$$
$$x(x-4)(x^2+4x+16)=0$$

a^3-b^3
$=(a-b)(a^2+ab+b^2)$

$$\therefore\ x=0,\ 4$$

Since $f(x)\leqslant g(x)$ in $0\leqslant x\leqslant 4$,

$$S=\int_0^4\left(\sqrt{x}-\frac{1}{8}x^2\right)dx$$
$$=\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{24}x^3\right]_0^4$$
$$=\frac{8}{3}$$



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Ex. Find the area S enclosed by two curves $y=\sin 2x$ and $y=\cos x$ in $0 \leqslant x \leqslant \pi$.

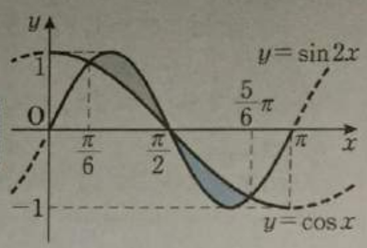
[Sol] When $\sin 2x=\cos x$,

$2\sin x\cos x-\cos x=0$

$\cos x(2\sin x-1)=0$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\therefore\ \cos x=0,\ \sin x=\frac{1}{2}$



Since $0 \leqslant x \leqslant \pi$, $x=\frac{\pi}{6}, \frac{\pi}{2}, \frac{5}{6}\pi$

$\sin 2x \geqslant \cos x$ in $\frac{\pi}{6} \leqslant x \leqslant \frac{\pi}{2}$ and $\sin 2x \leqslant \cos x$ in $\frac{\pi}{2} \leqslant x \leqslant \frac{5}{6}\pi$

$$\therefore\ S=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(\sin 2x-\cos x)dx+\int_{\frac{\pi}{2}}^{\frac{5}{6}\pi}(\cos x-\sin 2x)dx$$

$$=\left[-\frac{1}{2}\cos 2x-\sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}+\left[\sin x+\frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\frac{5}{6}\pi}=\frac{1}{2}$$

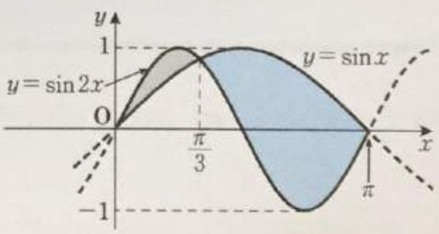
1. Find the area S enclosed by two curves $y=\sin x$ and $y=\sin 2x$ in $0 \leqslant x \leqslant \pi$.

[Sol] When $\sin x=\sin 2x$,

$\sin x-2\sin x\cos x=0$

$\sin x(1-2\cos x)=0$

$\therefore\ \sin x=0,\ \cos x=\frac{1}{2}$



Since $0 \leqslant x \leqslant \pi$, $x=0, \frac{\pi}{3}, \pi$

$\sin x \leqslant \sin 2x$ in $0 \leqslant x \leqslant \frac{\pi}{3}$ and $\sin x \geqslant \sin 2x$ in $\frac{\pi}{3} \leqslant x \leqslant \pi$

$$\therefore\ S=\int_{0}^{\frac{\pi}{3}}(\sin 2x-\sin x)dx+\int_{\frac{\pi}{3}}^{\pi}(\sin x-\sin 2x)dx$$

$$=\left[-\frac{1}{2}\cos 2x+\cos x\right]_{0}^{\frac{\pi}{3}}+\left[-\cos x+\frac{1}{2}\cos 2x\right]_{\frac{\pi}{3}}^{\pi}=\frac{5}{2}$$

O133b

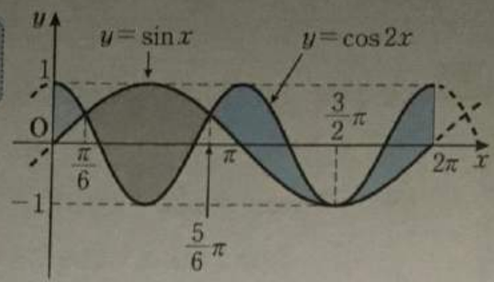
2. Find the area S enclosed by two curves $y=\sin x$ and $y=\cos 2x$ and two lines $x=0$ and $x=2\pi$ in $0 \leqslant x \leqslant 2\pi$.

[Sol] When $\sin x=\cos 2x$,

$2\sin^2 x+\sin x-1=0$ $\quad$ $\cos 2\alpha=1-2\sin^2\alpha$

$(\sin x+1)(2\sin x-1)=0$

$\therefore\ \sin x=-1,\ \frac{1}{2}$



Since $0 \leqslant x \leqslant 2\pi$, $x=\frac{\pi}{6},\ \frac{5}{6}\pi,\ \frac{3}{2}\pi$

$\sin x \leqslant \cos 2x$ in $0 \leqslant x \leqslant \frac{\pi}{6},\ \frac{5}{6}\pi \leqslant x \leqslant 2\pi$

$\sin x \geqslant \cos 2x$ in $\frac{\pi}{6} \leqslant x \leqslant \frac{5}{6}\pi$

At $x=\frac{3}{2}\pi$, $\sin x \leqslant \cos 2x$ for both before and after the point of intersection of the two curves

Therefore,

$$S=\int_0^{\frac{\pi}{6}}(\cos 2x-\sin x)dx+\int_{\frac{\pi}{6}}^{\frac{5}{6}\pi}(\sin x-\cos 2x)dx+\int_{\frac{5}{6}\pi}^{2\pi}(\cos 2x-\sin x)dx$$

$$=\left[\frac{1}{2}\sin 2x+\cos x\right]_0^{\frac{\pi}{6}}+\left[-\cos x-\frac{1}{2}\sin 2x\right]_{\frac{\pi}{6}}^{\frac{5}{6}\pi}+\left[\frac{1}{2}\sin 2x+\cos x\right]_{\frac{5}{6}\pi}^{2\pi}$$

$$=3\sqrt{3}$$

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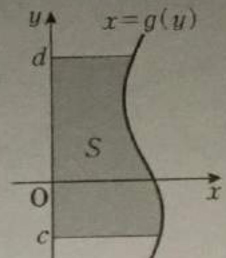
Given area S enclosed by the curve $x=g(y)$, the y-axis and two lines $y=c$ and $y=d$ $(c<d)$, the following formula is true.



Curve $x=g(y)$ and Area

On the interval $c \leqslant y \leqslant d$, when $g(y) \geqslant 0$,

$$S=\int_c^d g(y)dy$$

Ex. Find the area S enclosed by curve $y=\sqrt{x-1}$, the y-axis and lines $y=1$ and $y=2$.

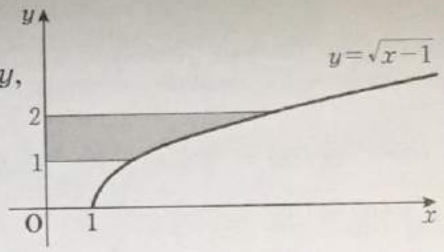
[Sol] Since $y=\sqrt{x-1}$, $x=y^2+1$

Since $y^2+1>0$ for all values of y,

$$S=\int_1^2 (y^2+1)dy$$

$$=\left[\frac{1}{3}y^3+y\right]_1^2$$

$$=\frac{10}{3}$$



1. Find the area S enclosed by curve $y=\frac{1}{\sqrt{x}}$, the y-axis and lines $y=\frac{1}{2}$ and $y=1$.

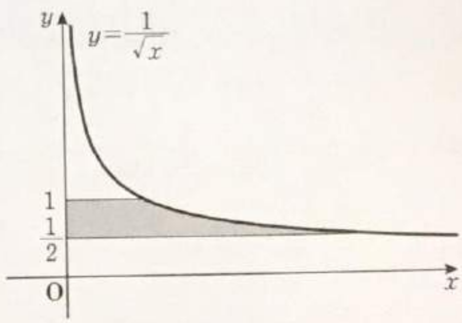
[Sol] Since $y=\frac{1}{\sqrt{x}}$, $x=\frac{1}{y^2}$

Since $\frac{1}{y^2}>0$ for all values of y,

$$S=\int_{\frac{1}{2}}^1 \frac{dy}{y^2}$$

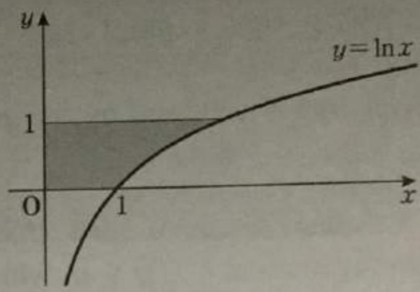
$$=\left[-\frac{1}{y}\right]_{\frac{1}{2}}^1$$

$$=1$$



O134b

2. Find the area S enclosed by curve $y=\ln x$, the x-axis, y-axis and line $y=1$.



[Sol] Since $y=\ln x$, $x=e^y$

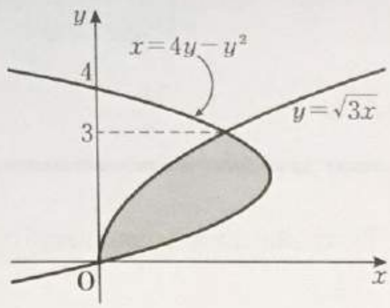
Since $e^y>0$ for all values of y,

$$S=\int_0^1 e^y\,dy$$

$$=\left[e^y\right]_0^1$$

$$=e-1$$

3. Find the area S enclosed by two curves $x=4y-y^2$ and $y=\sqrt{3x}$.



[Sol] Since $y=\sqrt{3x}$, $x=\frac{1}{3}y^2$

When $4y-y^2=\frac{1}{3}y^2$,

$$y^2-3y=0$$

$$y(y-3)=0$$

$$\therefore\ y=0,\ 3$$

Since $4y-y^2\geqslant\frac{1}{3}y^2$ in $0\leqslant y\leqslant 3$,

$$S=\int_0^3\left[(4y-y^2)-\frac{1}{3}y^2\right]dy$$

$$=\int_0^3\left(4y-\frac{4}{3}y^2\right)dy$$

$$=\left[2y^2-\frac{4}{9}y^3\right]_0^3$$

$$=6$$

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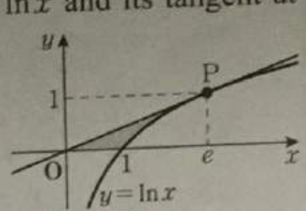
Ex. Find the area S enclosed by the x-axis, curve $y=\ln x$ and its tangent at point $\mathrm{P}(e, 1)$.



[Sol] Since $y'=\dfrac{1}{x}$, the equation of the tangent at point P is

$$y-1=\frac{1}{e}(x-e)$$

The equation of the tangent at $(a, f(a))$ is $y-f(a)=f'(a)(x-a)$

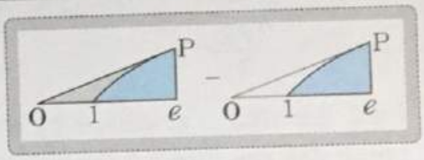
So, $y=\dfrac{1}{e}x$

When $\ln x=0$, $x=\boxed{1}$

When $\dfrac{1}{e}x=0$, $x=\boxed{0}$

Finding the x-coordinates of the points of intersection of the x-axis with each graph

Therefore,



$$S=\int_{\boxed{0}}^{e}\frac{1}{e}x\,dx-\int_{\boxed{1}}^{e}\ln x\,dx$$

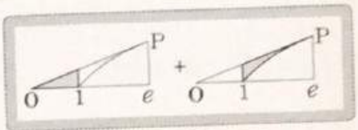
$$=\frac{1}{e}\left[\frac{1}{2}x^2\right]_{\boxed{0}}^{e}-\left(\left[x\ln x\right]_{\boxed{1}}^{e}-\int_{\boxed{1}}^{e}dx\right)$$

$$=\boxed{-\frac{1}{2}e}+\left[x\right]_{\boxed{1}}^{e}$$

$$=\boxed{\frac{1}{2}e-1}$$

Answers: 1, 0, 0, 1, 0, 1, 1, $-\frac{1}{2}e$, 1, $\frac{1}{2}e-1$

Ex. can also be considered as follows:



$$S=\int_0^1\frac{1}{e}x\,dx+\int_1^e\left(\frac{1}{e}x-\ln x\right)dx$$

$$=\int_0^e\frac{1}{e}x\,dx-\int_1^e\ln x\,dx$$

$$=\frac{1}{e}\left[\frac{1}{2}x^2\right]_0^e-\left(\left[x\ln x\right]_1^e-\int_1^e dx\right)$$

$$=-\frac{1}{2}e+\left[x\right]_1^e=\frac{1}{2}e-1$$

$\int_a^b f(x)dx+\int_b^c f(x)dx$
$=\int_a^c f(x)dx$

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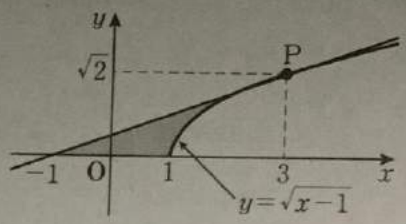
0135b

1. Find the area S enclosed by the x-axis, curve $y=\sqrt{x-1}$ and its tangent at point $P(3, \sqrt{2})$.

[Sol] Since $y'=\dfrac{1}{2\sqrt{x-1}}$, the equation of the tangent at point P is

$$y-\sqrt{2}=\frac{1}{2\sqrt{2}}(x-3)$$

So, $y=\dfrac{\sqrt{2}}{4}x+\dfrac{\sqrt{2}}{4}$



When $\sqrt{x-1}=0$, $x=1$

When $\dfrac{\sqrt{2}}{4}x+\dfrac{\sqrt{2}}{4}=0$, $x=-1$

Therefore,

$$S=\int_{-1}^{3}\left(\frac{\sqrt{2}}{4}x+\frac{\sqrt{2}}{4}\right)dx-\int_{1}^{3}\sqrt{x-1}\,dx$$

$$=\left[\frac{\sqrt{2}}{8}x^2+\frac{\sqrt{2}}{4}x\right]_{-1}^{3}-\left[\frac{2}{3}(x-1)^{\frac{3}{2}}\right]_{1}^{3}$$

$$=\frac{2\sqrt{2}}{3}$$

Alternative Solution

$$S=\int_{-1}^{1}\left(\frac{\sqrt{2}}{4}x+\frac{\sqrt{2}}{4}\right)dx+\int_{1}^{3}\left[\left(\frac{\sqrt{2}}{4}x+\frac{\sqrt{2}}{4}\right)-\sqrt{x-1}\right]dx$$

$$=\int_{-1}^{3}\left(\frac{\sqrt{2}}{4}x+\frac{\sqrt{2}}{4}\right)dx-\int_{1}^{3}\sqrt{x-1}\,dx$$

$$=\left[\frac{\sqrt{2}}{8}x^2+\frac{\sqrt{2}}{4}x\right]_{-1}^{3}-\left[\frac{2}{3}(x-1)^{\frac{3}{2}}\right]_{1}^{3}$$

$$=\frac{2\sqrt{2}}{3}$$

2. Find the area S enclosed by the y-axis, curve $y=e^{ax}$ and its tangent to the curve passing through the origin. (a is a positive constant.)

[Sol] Let the coordinates of the tangent point be (t, e^{at}).

Since $y'=ae^{ax}$, the equation of the tangent is

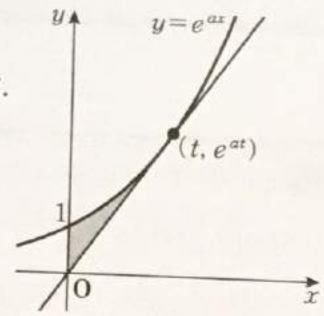
$$y-e^{at}=ae^{at}(x-t)$$

So, $y=ae^{at}x-e^{at}(at-1)$

Since this line passes through the origin, $0=-e^{at}(at-1)$

Since $e^{at}>0$, $t=\dfrac{1}{a}$

Therefore, the equation of the tangent is $y=aex$.



Thus,

$$S=\int_{0}^{\frac{1}{a}}(e^{ax}-aex)dx$$

$$=\left[\frac{1}{a}e^{ax}-\frac{1}{2}aex^2\right]_{0}^{\frac{1}{a}}$$

$$=\frac{e-2}{2a}$$

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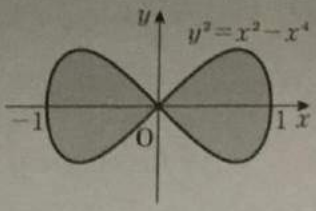
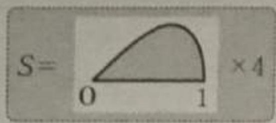
Ex. Curve $y^2=x^2-x^4$ is symmetric to both axes as shown on the right. Find the area S enclosed by this curve.



[Sol] Since $y^2=x^2-x^4$, $y=\pm x\sqrt{1-x^2}$

Since this curve is symmetric to both axes,

$$S=4\int_0^1 x\sqrt{1-x^2}\,dx$$



Let $\sqrt{1-x^2}=t$. Since $1-x^2=t^2$, $-2x\,dx=2t\,dt$

x	$0 \longrightarrow 1$
t	$1 \longrightarrow 0$

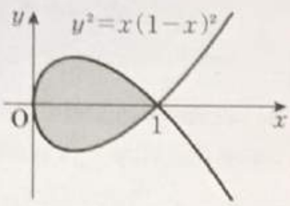
$$\therefore\ x\,dx=-t\,dt$$

$$\therefore\ S=4\int_1^0 t\cdot(-t)\,dt$$

$$=4\int_0^1 t^2\,dt$$

$$=4\left[\frac{1}{3}t^3\right]_0^1=\frac{4}{3}$$

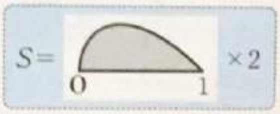
1. Curve $y^2=x(1-x)^2$ is symmetric to the x-axis as shown on the right. Find the area S enclosed by this curve.



[Sol] Since $y^2=x(1-x)^2$, $y=\pm\sqrt{x}(1-x)$

Since this curve is symmetric to the x-axis,

$$S=2\int_0^1 \sqrt{x}(1-x)\,dx$$



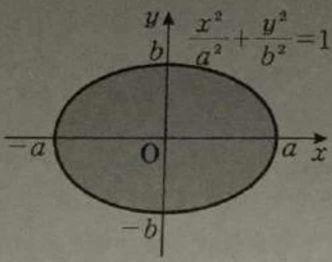
$$\therefore\ S=2\int_0^1 (\sqrt{x}-x\sqrt{x})\,dx$$

$$=2\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{2}{5}x^{\frac{5}{2}}\right]_0^1=\frac{8}{15}$$

If the equation is the same even after y is replaced by $-y$, the graph is symmetric with respect to the x-axis. Likewise, if the equation is the same even after x is replaced by $-x$, the graph is symmetric with respect to the y-axis.

0136b

2. Find the area S enclosed by ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
$(a>0,\ b>0)$

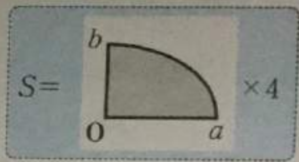


[Sol] Since $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $y^2=\frac{b^2}{a^2}(a^2-x^2)$

$\therefore\ y=\pm\frac{b}{a}\sqrt{a^2-x^2}$

Since this ellipse is symmetric to both axes,

$$S=4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$



Let $x=a\sin\theta$. $dx=a\cos\theta\,d\theta$ ← 0105

Since $\cos\theta\geqslant 0$ in $0\leqslant\theta\leqslant\frac{\pi}{2}$,

$$\sqrt{a^2-x^2}=\sqrt{a^2(1-\sin^2\theta)}=a\cos\theta$$

x	$0 \longrightarrow a$
θ	$0 \longrightarrow \frac{\pi}{2}$

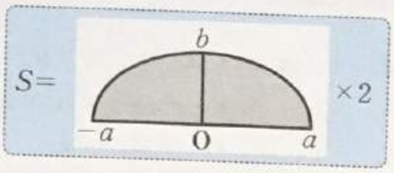
Therefore,

$$S=4\int_0^{\frac{\pi}{2}}\frac{b}{a}\cdot a\cos\theta\cdot a\cos\theta\,d\theta$$
$$=4ab\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$
$$=2ab\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta$$
$$=2ab\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}=\boldsymbol{\pi ab}$$

Alternative Solution

Since this ellipse is symmetric to the x-axis,

$$S=2\int_{-a}^a\frac{b}{a}\sqrt{a^2-x^2}\,dx$$



Let $x=a\sin\theta$. $dx=a\cos\theta\,d\theta$

Since $\cos\theta\geqslant 0$ in $-\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}$,

$$\sqrt{a^2-x^2}=\sqrt{a^2(1-\sin^2\theta)}=a\cos\theta$$

x	$-a \longrightarrow a$
θ	$-\frac{\pi}{2} \longrightarrow \frac{\pi}{2}$

$$\therefore\ S=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{b}{a}\cdot a\cos\theta\cdot a\cos\theta\,d\theta=2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$
$$=ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta=ab\left[\theta+\frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\boldsymbol{\pi ab}$$

Areas

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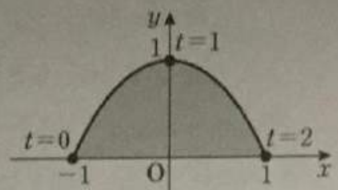
Ex. Find the area enclosed by the x-axis and the curve represented by $x=t-1,\ y=2t-t^2$ in $0 \leqq t \leqq 2$.

[Sol] When $y=0$, since $t(2-t)=0,\ t=0,\ 2$

Also, $y \geqq 0$ in $0 \leqq t \leqq 2$.

Then, since $x=t-1,\ dx=dt$

Therefore, let S be the area to be found.



x	$-1 \longrightarrow 1$
t	$0 \longrightarrow 2$

$$S=\int_{-1}^{1} y\,dx$$

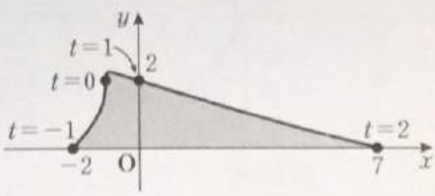
$$=\int_{0}^{2}(2t-t^2)\,dt$$

$$=\left[t^2-\frac{1}{3}t^3\right]_0^2=\frac{4}{3}$$

1. Find the area enclosed by the x-axis and the curve represented by $x=t^3-1$, $y=2+t-t^2$ in $-1 \leqq t \leqq 2$.

[Sol] When $y=0$, since $(t+1)(t-2)=0$,

$t=-1,\ 2$



Also, $y \geqq 0$ in $-1 \leqq t \leqq 2$.

Then, since $x=t^3-1,\ dx=3t^2\,dt$

Therefore, let S be the area to be found.

x	$-2 \longrightarrow 7$
t	$-1 \longrightarrow 2$

$$S=\int_{-2}^{7} y\,dx$$

$$=\int_{-1}^{2}(2+t-t^2)\cdot 3t^2\,dt$$

$$=3\int_{-1}^{2}(2t^2+t^3-t^4)\,dt$$

$$=3\left[\frac{2}{3}t^3+\frac{1}{4}t^4-\frac{1}{5}t^5\right]_{-1}^{2}=\mathbf{\frac{189}{20}}$$

O137b

2. Find the area enclosed by the x-axis and the curve represented by
$x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ $(a>0)$ in $0\leqslant\theta\leqslant 2\pi$.

[Sol] When $y=0$ in $0\leqslant\theta\leqslant 2\pi$,

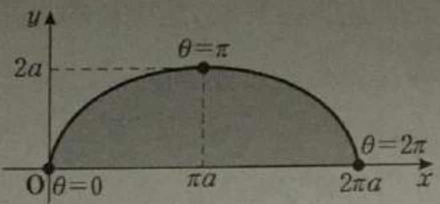
since $1-\cos\theta=0$, $\cos\theta=1$

$\therefore\ \theta=0,\ 2\pi$

Also, $y\geqslant 0$ in $0\leqslant\theta\leqslant 2\pi$.

Then, since $x=a(\theta-\sin\theta)$,

$dx=a(1-\cos\theta)d\theta$



x	$0 \longrightarrow 2\pi a$
θ	$0 \longrightarrow 2\pi$

Therefore, let S be the area to be found.

$$S=\int_0^{2\pi a} y\,dx$$

$$=\int_0^{2\pi} a(1-\cos\theta)\cdot a(1-\cos\theta)d\theta$$

$$=a^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)d\theta$$

$$=a^2\int_0^{2\pi}\left[1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$=a^2\int_0^{2\pi}\left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$=a^2\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$=3\pi a^2$$

The curve shown in question 2 is called a ***cycloid***.

O138a KUMON O 138

Areas

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1. Find the area enclosed by curve $2x^2-2xy+y^2=4$.

[Sol] Since $y^2-2xy+2x^2-4=0$,

$$y=x\pm\sqrt{x^2-(2x^2-4)}=x\pm\sqrt{4-x^2}$$

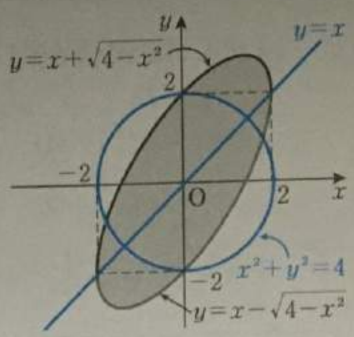
Therefore, let S be the area to be found.

$$S=\int_{-2}^{2}[(x+\sqrt{4-x^2})-(x-\sqrt{4-x^2})]dx$$

$$=2\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

$$=4\int_{0}^{2}\sqrt{4-x^2}\,dx$$

Even function



The sketch of the graph can be drawn by adding $y=x$ and $y=\pm\sqrt{4-x^2}$ (i.e. $x^2+y^2=4$).

Let $x=2\sin\theta$. $dx=2\cos\theta d\theta$ ← O105

Since $\cos\theta\geqq 0$ in $0\leqq\theta\leqq\frac{\pi}{2}$,

$$\sqrt{4-x^2}=\sqrt{4(1-\sin^2\theta)}=2\cos\theta$$

x	$0 \longrightarrow 2$
θ	$0 \longrightarrow \frac{\pi}{2}$

Thus,

$$S=4\int_{0}^{\frac{\pi}{2}}2\cos\theta\cdot 2\cos\theta d\theta$$

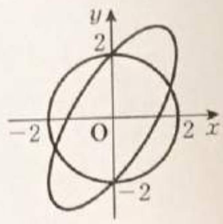
$$=16\int_{0}^{\frac{\pi}{2}}\cos^2\theta d\theta$$

$$=8\int_{0}^{\frac{\pi}{2}}(1+\cos 2\theta)d\theta$$

$$=8\left[\theta+\frac{1}{2}\sin 2\theta\right]_{0}^{\frac{\pi}{2}}$$

$$=4\pi$$

The value found in question 1 is equal to the area of a circle with radius 2 (i.e. $x^2+y^2=4$).



0138b

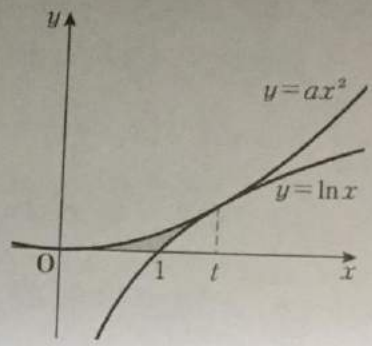
2. Find the value of the positive number a for which curves $y=\ln x$ and $y=ax^2$ touch at only one point. Then, find the area enclosed by the x-axis and the two curves.

[Sol] Let $f(x)=\ln x$ and $g(x)=ax^2$. $f'(x)=\frac{1}{x}$, $g'(x)=2ax$

Let t $(t>0)$ be the x-coordinate of the tangent point of $y=f(x)$ and $y=g(x)$. ⟵ Antilogarithm >0

$\ln t=at^2$ ⋯①

$\frac{1}{t}=2at$ ⋯② ⟵ 06



From ②, $a=\frac{1}{2t^2}$ ⋯③

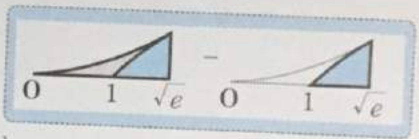
From ① and ③, $\ln t=\frac{1}{2}$

$\therefore\ t=\sqrt{e}$

$\therefore\ a=\frac{1}{2e}$

Therefore, let S be the area to be found.

$$S=\int_0^{\sqrt{e}}\frac{1}{2e}x^2dx-\int_1^{\sqrt{e}}\ln x\,dx$$ ⟵



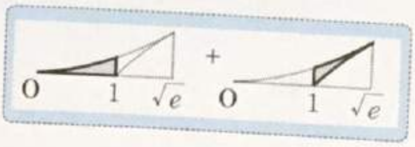
$$=\frac{1}{2e}\left[\frac{1}{3}x^3\right]_0^{\sqrt{e}}-\left(\left[x\ln x\right]_1^{\sqrt{e}}-\int_1^{\sqrt{e}}dx\right)$$

$$=-\frac{\sqrt{e}}{3}+\left[x\right]_1^{\sqrt{e}}$$

$$=\frac{2\sqrt{e}}{3}-1$$

Alternative Solution

$$S=\int_0^1\frac{1}{2e}x^2dx+\int_1^{\sqrt{e}}\left(\frac{1}{2e}x^2-\ln x\right)dx$$ ⟵



$$=\int_0^{\sqrt{e}}\frac{1}{2e}x^2dx-\int_1^{\sqrt{e}}\ln x\,dx$$

$$=\frac{1}{2e}\left[\frac{1}{3}x^3\right]_0^{\sqrt{e}}-\left(\left[x\ln x\right]_1^{\sqrt{e}}-\int_1^{\sqrt{e}}dx\right)$$

$$=-\frac{\sqrt{e}}{3}+\left[x\right]_1^{\sqrt{e}}$$

$$=\frac{2\sqrt{e}}{3}-1$$

Areas

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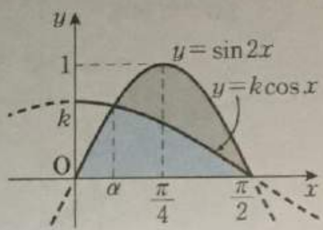
1. Given two curves $C_1 : y=\sin 2x$ and $C_2 : y=k\cos x$ $\left(0 \leqslant x \leqslant \frac{\pi}{2},\ 0 \leqslant k \leqslant \frac{\pi}{2}\right)$, find the value of constant k for which C_2 bisects the area enclosed by C_1 and the x-axis.



[Sol] The points of intersection of C_1 and the x-axis are $x=0,\ \frac{\pi}{2}$. Since $\sin 2x \geqslant 0$ in $0 \leqslant x \leqslant \frac{\pi}{2}$, the area enclosed by curve C_1 and the x-axis is

$$\int_0^{\frac{\pi}{2}} \sin 2x\,dx = \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}} = 1 \quad \cdots ①$$

Since $\sin 2x = k\cos x$, the x-coordinates of the points of intersection of two curves C_1 and C_2 are

$$2\sin x\cos x - k\cos x = 0$$ $\sin 2x = 2\sin x\cos x$

$$\cos x(2\sin x - k) = 0$$

$$\therefore\ \cos x = 0,\ \sin x = \frac{k}{2}$$

Since $0 \leqslant x \leqslant \frac{\pi}{2}$, $x=\alpha,\ \frac{\pi}{2}$. However, $\sin\alpha = \frac{k}{2}$ $\left(0 < \alpha < \frac{\pi}{2}\right)$

Since $\sin 2x \geqslant k\cos x$ in $\alpha \leqslant x \leqslant \frac{\pi}{2}$,

the area enclosed by two curves C_1 and C_2 is

$$\int_\alpha^{\frac{\pi}{2}} (\sin 2x - k\cos x)\,dx = \left[-\frac{1}{2}\cos 2x - k\sin x\right]_\alpha^{\frac{\pi}{2}}$$

$$= \frac{1}{2} - k + \frac{1}{2}\cos 2\alpha + k\sin\alpha$$

$$= \frac{1}{2} - k + \frac{1}{2}(1 - 2\sin^2\alpha) + k\sin\alpha$$ $\cos 2\alpha = 1 - 2\sin^2\alpha$

$$= \frac{1}{4}k^2 - k + 1 \quad \cdots ②$$ $\sin\alpha = \frac{k}{2}$

From ① and ②, $\frac{1}{4}k^2 - k + 1 = \frac{1}{2}$ — Since the area, enclosed by C_1 and the x-axis, is bisected by C_2

$$k^2 - 4k + 2 = 0$$ From Quadratic Formula II (J102), $k = 2 \pm \sqrt{(-2)^2 - 1\cdot 2}$

$$\therefore\ k = 2 \pm \sqrt{2}$$

Since $0 \leqslant k \leqslant \frac{\pi}{2}$, $k = 2 - \sqrt{2}$ — $2-\sqrt{2} \doteqdot 0.59,\ 2+\sqrt{2} \doteqdot 3.41$, $\frac{\pi}{2} \doteqdot 1.57$

0139b

2. Given curve $y=e^{-x}\sin x$ $(x \geqq 0)$, let the areas enclosed by the curve and the x-axis (above the x-axis) be S_0, S_1, ..., S_n, ... in order from the y-axis. Find $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n} S_k$.

[Sol] Since $e^{-x}\sin x=0$, the points of intersection of curve $y=e^{-x}\sin x$ $(x \geqq 0)$ and the x-axis are $\sin x=0$, i.e. $x=n\pi$ $(n=0,\ 1,\ 2,\ \dots)$.

Also, since $e^{-x}>0$, $y \geqq 0$ when $\sin x \geqq 0$.

Therefore,

$y \geqq 0$ when $2n\pi \leqq x \leqq (2n+1)\pi$

The intervals of the areas to be found are $0 \leqq x \leqq \pi$, $2\pi \leqq x \leqq 3\pi$, ...; i.e. the lower limits are even multiples of π and the upper limits are odd multiples of π.

$$\therefore\ \boxed{S_k} = \int_{2k\pi}^{(2k+1)\pi} e^{-x}\sin x\,dx$$

$$= \Big[-e^{-x}\sin x\Big]_{2k\pi}^{(2k+1)\pi} + \int_{2k\pi}^{(2k+1)\pi} e^{-x}\cos x\,dx \qquad ※1$$

$$= \Big[-e^{-x}\cos x\Big]_{2k\pi}^{(2k+1)\pi} - \int_{2k\pi}^{(2k+1)\pi} e^{-x}\sin x\,dx \qquad ※1\ ※2$$

$$= e^{-(2k+1)\pi}+e^{-2k\pi}-\boxed{S_k} \qquad ※2$$

$$\therefore\ S_k=\frac{1}{2}[e^{-(2k+1)\pi}+e^{-2k\pi}]$$

※1 $\int f'(x)g(x)dx=f(x)g(x)-\int f(x)g'(x)dx$

※2 When k is an integer,
$\sin(2k+1)\pi=0,\ \sin 2k\pi=0$
$\cos(2k+1)\pi=-1,\ \cos 2k\pi=1$

Thus,

$$\sum_{k=0}^{n} S_k=\sum_{k=0}^{n}\left\{\frac{1}{2}[e^{-(2k+1)\pi}+e^{-2k\pi}]\right\}$$

$$=\frac{1}{2}\left[\sum_{k=0}^{n} e^{-(2k+1)\pi}+\sum_{k=0}^{n} e^{-2k\pi}\right]$$

$$=\frac{1}{2}\left\{\frac{e^{-\pi}[1-e^{-2(n+1)\pi}]}{1-e^{-2\pi}}+\frac{1-e^{-2(n+1)\pi}}{1-e^{-2\pi}}\right\}$$

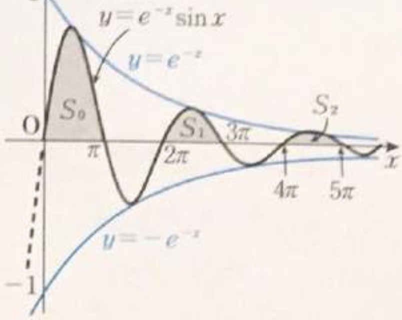
$\sum\limits_{k=0}^{n} e^{-(2k+1)\pi}$, $\sum\limits_{k=0}^{n} e^{-2k\pi}$ are the sums of the geometric sequences with 1st term $e^{-\pi}$ and common ratio $e^{-2\pi}$, and 1st term 1 and common ratio $e^{-2\pi}$ respectively. (The numbers of terms are $n+1$.)

Therefore,

$$\lim_{n\to\infty}\sum_{k=0}^{n} S_k=\frac{1}{2}\cdot\frac{e^{-\pi}+1}{1-e^{-2\pi}} \qquad \lim_{n\to\infty} e^{-2(n+1)\pi}=0$$

$$=\frac{1+e^{-\pi}}{2(1+e^{-\pi})(1-e^{-\pi})}$$

$$=\frac{1}{2(1-e^{-\pi})}\quad \left[=\frac{e^{\pi}}{2(e^{\pi}-1)}\right]$$



O140a

KUMON

Areas

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Time : to :

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1. Find the area S enclosed by the following curves and lines.

➡ O132

(1) $y=x,\quad y=\sqrt{x}$

[Sol] When $x=\sqrt{x}$,

$$x^2-x=0$$

$$x(x-1)=0$$

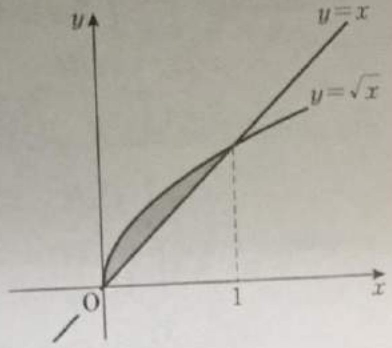
$$\therefore\ x=0,\ 1$$

Since $x \leqslant \sqrt{x}$ in $0 \leqslant x \leqslant 1$,

$$S=\int_0^1(\sqrt{x}-x)dx$$

$$=\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{2}x^2\right]_0^1$$

$$=\frac{1}{6}$$



➡ O134

(2) $y^2=x,\quad x+y-6=0$

[Sol] Since $x+y-6=0$, $x=-y+6$

When $y^2=-y+6$,

$$y^2+y-6=0$$

$$(y+3)(y-2)=0$$

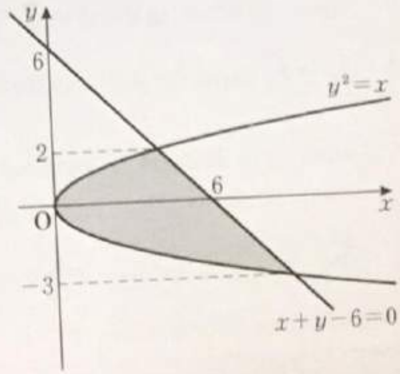
$$\therefore\ y=-3,\ 2$$

Since $y^2 \leqslant -y+6$ in $-3 \leqslant y \leqslant 2$,

$$S=\int_{-3}^2[(-y+6)-y^2]dy$$

$$=\left[-\frac{1}{2}y^2+6y-\frac{1}{3}y^3\right]_{-3}^2$$

$$=\frac{125}{6}$$



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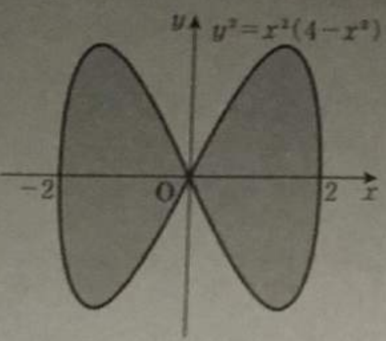
0140b

2. Curve $y^2=x^2(4-x^2)$ is symmetric to both axes as shown on the right. Find the area S enclosed by this curve.
➡ 0136



[Sol] Since $y^2=x^2(4-x^2)$, $y=\pm x\sqrt{4-x^2}$

Since this curve is symmetric to both axes,

$$S=4\int_0^2 x\sqrt{4-x^2}\,dx$$

Let $\sqrt{4-x^2}=t$. Since $4-x^2=t^2$, $-2x\,dx=2t\,dt$

$$\therefore\ x\,dx=-t\,dt$$

x	$0 \longrightarrow 2$
t	$2 \longrightarrow 0$

$$\therefore\ S=4\int_2^0 t\cdot(-t)\,dt$$
$$=4\int_0^2 t^2\,dt$$
$$=4\left[\frac{1}{3}t^3\right]_0^2$$
$$=\frac{32}{3}$$

Alternative Solution 1

Let $4-x^2=t$. $-2x\,dx=dt$

x	$0 \longrightarrow 2$
t	$4 \longrightarrow 0$

$$S=4\int_4^0 \sqrt{t}\cdot\left(-\frac{1}{2}\right)dt=2\int_0^4 t^{\frac{1}{2}}dt=2\left[\frac{2}{3}t^{\frac{3}{2}}\right]_0^4=\frac{32}{3}$$

Alternative Solution 2

Let $x=2\sin\theta$. $dx=2\cos\theta\,d\theta$

x	$0 \longrightarrow 2$
θ	$0 \longrightarrow \frac{\pi}{2}$

Since $\cos\theta\geqslant 0$ in $0\leqslant\theta\leqslant\frac{\pi}{2}$, $\sqrt{4-x^2}=\sqrt{4(1-\sin^2\theta)}=2\cos\theta$

$$S=4\int_0^{\frac{\pi}{2}} 2\sin\theta\cdot 2\cos\theta\cdot 2\cos\theta\,d\theta=32\int_0^{\frac{\pi}{2}}\sin\theta\cos^2\theta\,d\theta=32\left[-\frac{1}{3}\cos^3\theta\right]_0^{\frac{\pi}{2}}=\frac{32}{3}$$

Alternative Solution 3 (without using the symmetric property)

$$S=\int_{-2}^0(-x\sqrt{4-x^2}-x\sqrt{4-x^2})dx+\int_0^2[x\sqrt{4-x^2}-(-x\sqrt{4-x^2})]dx$$
$$=-2\int_{-2}^0 x\sqrt{4-x^2}\,dx+2\int_0^2 x\sqrt{4-x^2}\,dx$$

Let $\sqrt{4-x^2}=t$. Since $4-x^2=t^2$, $-2x\,dx=2t\,dt$

$\therefore\ x\,dx=-t\,dt$

x	$-2 \longrightarrow 0$
t	$0 \longrightarrow 2$

x	$0 \longrightarrow 2$
t	$2 \longrightarrow 0$

$$S=-2\int_0^2 t\cdot(-t)\,dt+2\int_2^0 t\cdot(-t)\,dt=2\int_0^2 t^2dt+2\int_0^2 t^2dt=4\int_0^2 t^2dt=4\left[\frac{1}{3}t^3\right]_0^2=\frac{32}{3}$$

Volumes

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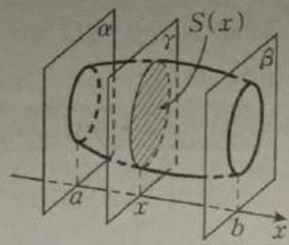
As shown in the diagram, a solid is held between two planes α and β which are perpendicular to the x-axis. Let V be the volume of the solid, and a and b be the coordinates of the points of intersection of the x-axis with two planes α and β respectively $(a < b)$.

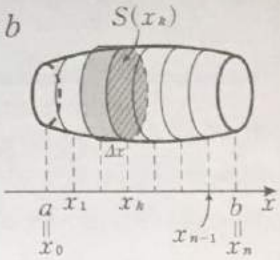


In $a \leqslant x \leqslant b$, let $S(x)$ be the cross-sectional area of the solid cut by plane γ which is perpendicular to the x-axis and intersecting the x-axis at x. Find the volume V of the solid.

[Sol] Dividing the interval $[a, b]$ into n equal subintervals, let both boundaries and dividing points be

$$a=x_0,\ x_1,\ x_2,\ \dots,\ x_{k-1},\ x_k,\ \dots,\ x_{n-1},\ x_n=b$$

Then, let $\Delta x=\dfrac{b-a}{n}$.



Divide the solid by the planes which pass through each dividing point and are perpendicular to the x-axis.

Consider each of the n solids as a thick slice of solid with cross-sectional area $S(x_k)$ and width Δx.

Let the sum of the volumes be V_n.

$$V_n=S(x_1)\Delta x+S(x_2)\Delta x+\cdots+S(x_n)\Delta x$$

$$=\sum_{k=1}^{n}\boxed{S(x_k)\Delta x}$$

If n approaches infinity, V_n approaches V.

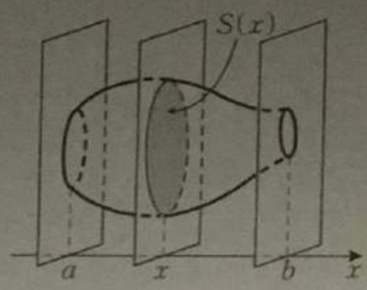
$$\therefore\ V=\lim_{n\to\infty}\sum_{k=1}^{n}\boxed{S(x_k)\Delta x}=\int_a^b S(x)\,dx$$

Definite Integrals and Limits of Sums I (O122)

Answers: All the answers are the same, $S(x_k)\Delta x$

0141b

From the result on side a, let $S(x)$ be the cross-sectional area of a solid cut by the plane perpendicular to the x-axis. Then, the following formula regarding volume V in $a \leqslant x \leqslant b$ is true.

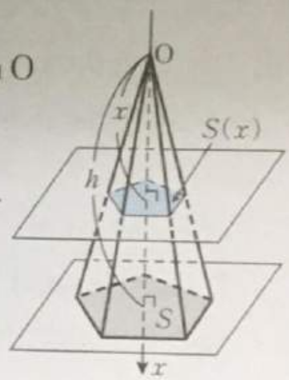


Cross-Sectional Area and Volume of the Solid

$$V=\int_a^b S(x)dx$$

Ex. Using integration, find the volume V of the pyramid with base area S and height h as shown in the diagram.

[Sol] Let the vertex be O and the perpendicular from O to the base be the x-axis.
In $0 \leqslant x \leqslant h$, let $S(x)$ be the cross-sectional area of the solid cut by the plane perpendicular to the x-axis and intersecting the x-axis at x.



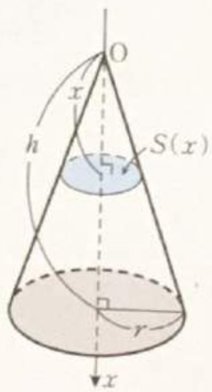
$S(x):S=x^2:h^2$ ⟵ If the ratio of the length is $m:n$, the ratio of the area is $m^2:n^2$.

$\therefore\ S(x)=\dfrac{S}{h^2}x^2$

$\therefore\ V=\displaystyle\int_0^h \frac{S}{h^2}x^2\,dx=\frac{S}{h^2}\left[\frac{1}{3}x^3\right]_0^h=\frac{1}{3}Sh$ ⟵ $V=\displaystyle\int_a^b S(x)dx$

1. Using integration, find the volume V of the cone with base radius r and height h as shown in the diagram.

[Sol] Let the vertex be O and the perpendicular from O to the base be the x-axis.
In $0 \leqslant x \leqslant h$, let $S(x)$ be the cross-sectional area of the solid cut by the plane perpendicular to the x-axis and intersecting the x-axis at x, and let the base area be S.



$S(x):S=x^2:h^2$

Since $S=\pi r^2$, $S(x)=\dfrac{\pi r^2}{h^2}x^2$

$\therefore\ V=\displaystyle\int_0^h \frac{\pi r^2}{h^2}x^2dx=\frac{\pi r^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h=\frac{1}{3}\pi r^2 h$

Volumes

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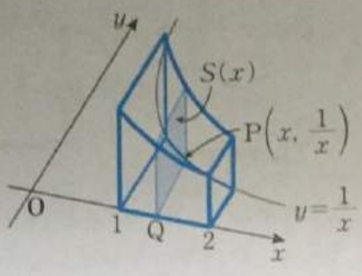
Ex. Given that point $\mathrm{P}\left(x, \dfrac{1}{x}\right)$ is on curve $y=\dfrac{1}{x}$ $(x>0)$, let Q be the point of intersection of the x-axis and the line parallel to the y-axis passing through point P. Place a square with side PQ on the plane perpendicular to the x-axis. Find the volume V formed by the square as the x-coordinate of point P moves from 1 to 2.

[Sol] $\mathrm{PQ}=\dfrac{1}{x}$

Let the area of the square be $S(x)$. $S(x)=\dfrac{1}{x^2}$

$$\therefore\ V=\int_1^2 \frac{dx}{x^2}=\left[-\frac{1}{x}\right]_1^2=\frac{1}{2}$$

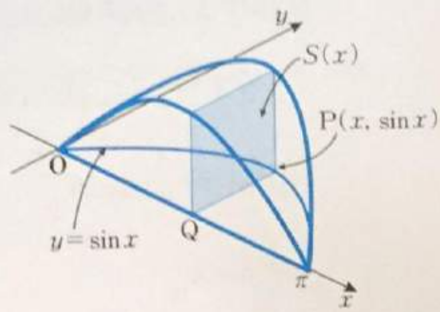


1. Given that point $\mathrm{P}(x, \sin x)$ is on curve $y=\sin x$ $(0 \leqslant x \leqslant \pi)$, let Q be the point of intersection of the x-axis and the line parallel to the y-axis passing through point P. Place a square with side PQ on the plane perpendicular to the x-axis. Find the volume V formed by the square as the x-coordinate of point P moves from 0 to π.

[Sol] $\mathrm{PQ}=\sin x$

Let the area of the square be $S(x)$. $S(x)=\sin^2 x$

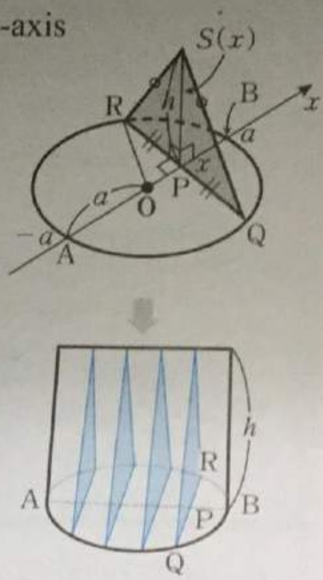
$$\therefore\ V=\int_0^{\pi} \sin^2 x\, dx$$
$$=\frac{1}{2}\int_0^{\pi}(1-\cos 2x)dx$$
$$=\frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^{\pi}$$
$$=\frac{\pi}{2}$$



O142b

2. Given that point P is on diameter AB of a circle with center O and radius a, let QR be the chord perpendicular to AB that passes through point P. Place an isosceles triangle with base QR and height h on the plane perpendicular to circle O. Find the volume V formed by the triangle as point P moves from A to B.

[Sol] Let the center of circle O be the origin, AB be the x-axis and the coordinate of point P be x.



Let the area of the isosceles triangle with base QR be $S(x)$.

Since $\mathrm{PR}=\sqrt{a^2-x^2}$,

$$S(x)=\frac{1}{2}\mathrm{QR}\cdot h=\frac{1}{2}\cdot 2\mathrm{PR}\cdot h=h\sqrt{a^2-x^2}$$

Therefore,

$$V=\int_{-a}^{a} h\sqrt{a^2-x^2}\,dx$$

$$=2h\int_{0}^{a}\sqrt{a^2-x^2}\,dx \quad \text{Even function}$$

Let $x=a\sin\theta$. $dx=a\cos\theta\,d\theta$ ⟵ O105

x	$0 \longrightarrow a$
θ	$0 \longrightarrow \frac{\pi}{2}$

When $0\leqslant\theta\leqslant\frac{\pi}{2}$, $\cos\theta\geqslant 0$; therefore,

$$\sqrt{a^2-x^2}=\sqrt{a^2(1-\sin^2\theta)}=a\cos\theta$$

Thus,

$$V=2h\int_{0}^{\frac{\pi}{2}} a\cos\theta\cdot a\cos\theta\,d\theta$$

$$=2a^2h\int_{0}^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$$=a^2h\int_{0}^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta$$

$$=a^2h\left[\theta+\frac{1}{2}\sin 2\theta\right]_{0}^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\pi a^2h$$

Volumes

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Time : to :

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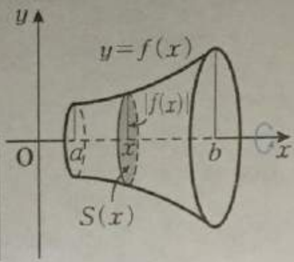
Let V be the volume of the solid formed by rotating the area enclosed by the curve $y=f(x)$, the x-axis and two lines $x=a$ and $x=b$ $(a<b)$ once about the x-axis. Cutting the solid by the plane which is perpendicular to the x-axis and passes through point $(x, 0)$, the cross-sectional area $S(x)$ is

$$S(x)=\pi|f(x)|^2=\pi[f(x)]^2$$

Therefore, the following formula is true.



Volume of Revolution about the x-axis

$$V=\pi\int_a^b [f(x)]^2\,dx=\pi\int_a^b y^2\,dx$$

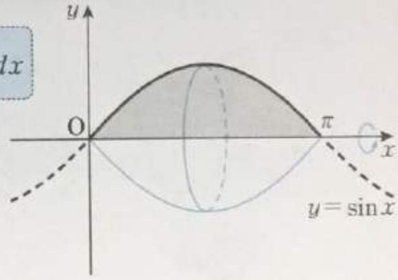
Ex. Given a solid formed by rotating the area enclosed by curve $y=\sin x$ $(0 \leqslant x \leqslant \pi)$ and the x-axis once about the x-axis, find the volume V of this solid.

[Sol] $V=\pi\int_0^{\pi} \sin^2 x\,dx \longleftarrow \boxed{V=\pi\int_a^b y^2\,dx}$

$$=\frac{\pi}{2}\int_0^{\pi}(1-\cos 2x)\,dx$$

$$=\frac{\pi}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^{\pi}$$

$$=\frac{\pi^2}{2}$$



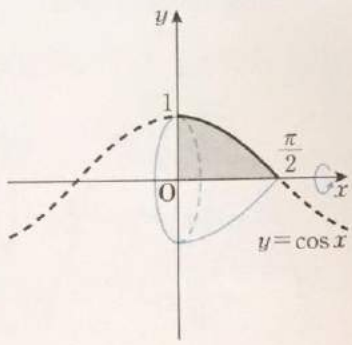
1. Given a solid formed by rotating the area enclosed by curve $y=\cos x$ $\left(0 \leqslant x \leqslant \frac{\pi}{2}\right)$, the x-axis and the y-axis once about the x-axis, find the volume V of this solid.

[Sol] $V=\pi\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}(1+\cos 2x)\,dx$$

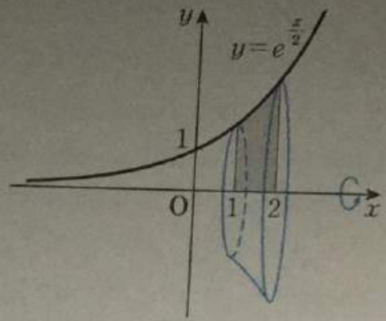
$$=\frac{\pi}{2}\left[x+\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi^2}{4}$$



O143b

2. Given a solid formed by rotating the area enclosed by curve $y=e^{\frac{x}{2}}$, the x-axis and two lines $x=1$ and $x=2$ once about the x-axis, find the volume V of this solid.

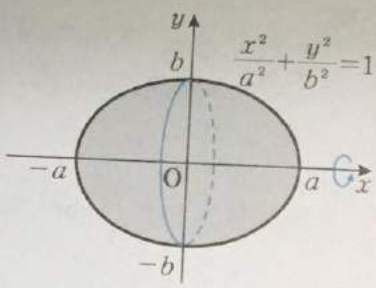


[Sol] $V=\pi\int_1^2 e^x\,dx$

$=\pi\Big[e^x\Big]_1^2$

$=\pi e(e-1)$

3. Let $a>0$ and $b>0$. Given a solid formed by rotating the area enclosed by ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ once about the x-axis, find the volume V of this solid.



[Sol] Since $y^2=\frac{b^2}{a^2}(a^2-x^2)$,

$V=\pi\int_{-a}^{a}\frac{b^2}{a^2}(a^2-x^2)dx$

$=\frac{2\pi b^2}{a^2}\int_0^a(a^2-x^2)dx$ Even function

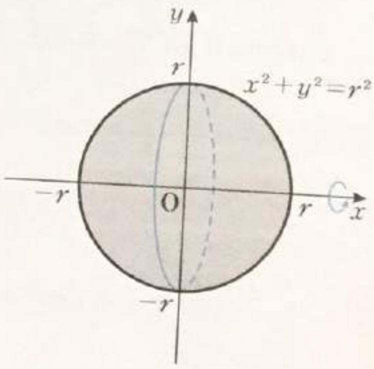
$=\frac{2\pi b^2}{a^2}\Big[a^2x-\frac{1}{3}x^3\Big]_0^a$

$=\frac{4}{3}\pi ab^2$

4. Using integration, find the volume V of the sphere with radius r.

[Sol] A sphere with radius r can be obtained by rotating a circle centered at origin O with radius r once about the x-axis.

Since $y^2=r^2-x^2$,



$V=\pi\int_{-r}^{r}(r^2-x^2)dx$

$=2\pi\int_0^r(r^2-x^2)dx$ Even function

$=2\pi\Big[r^2x-\frac{1}{3}x^3\Big]_0^r$

$=\frac{4}{3}\pi r^3$

Volumes

Name

Date / /

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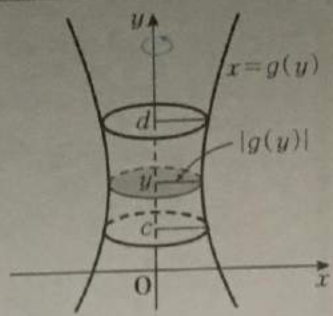
Time : to :

Let V be the volume of the solid formed by rotating the area enclosed by the curve $x=g(y)$, the y-axis and two lines $y=c$ and $y=d$ $(c<d)$ once about the y-axis. As with the case when rotating an area once about the x-axis, the following formula is true.



Volume of Revolution about the y-axis

$$V=\pi\int_c^d [g(y)]^2\,dy=\pi\int_c^d x^2\,dy$$

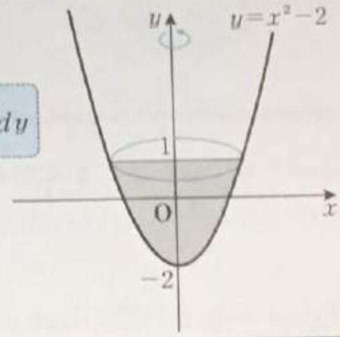
Ex. Given a solid formed by rotating the area enclosed by curve $y=x^2-2$ and line $y=1$ once about the y-axis, find the volume V of this solid.

[Sol] Since $x^2=y+2$,

$$V=\pi\int_{-2}^{1}(y+2)\,dy \quad \Leftarrow \quad \boxed{V=\pi\int_c^d x^2\,dy}$$

$$=\pi\left[\frac{1}{2}y^2+2y\right]_{-2}^{1}$$

$$=\frac{9}{2}\pi$$



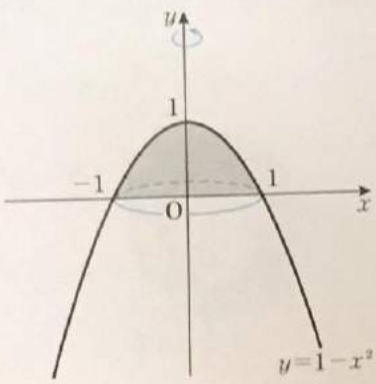
1. Given a solid formed by rotating the area enclosed by curve $y=1-x^2$ and the x-axis once about the y-axis, find the volume V of this solid.

[Sol] Since $x^2=1-y$,

$$V=\pi\int_0^1(1-y)\,dy$$

$$=\pi\left[y-\frac{1}{2}y^2\right]_0^1$$

$$=\frac{\pi}{2}$$

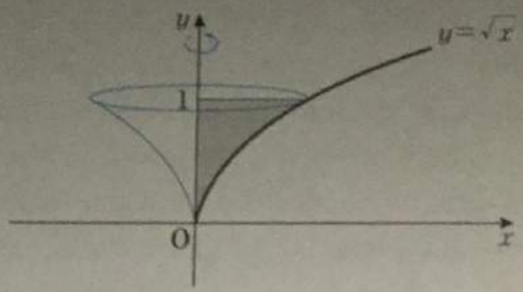


0144b

2. Given a solid formed by rotating the area enclosed by curve $y=\sqrt{x}$, the y-axis and line $y=1$ once about the y-axis, find the volume V of this solid.

[Sol] Since $x=y^2$,

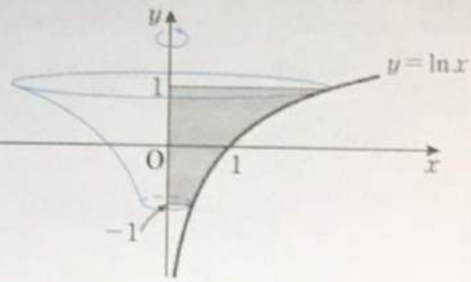
$$V=\pi\int_0^1 y^4\,dy$$
$$=\pi\left[\frac{1}{5}y^5\right]_0^1$$
$$=\frac{\pi}{5}$$



3. Given a solid formed by rotating the area enclosed by curve $y=\ln x$, the y-axis and two lines $y=-1$ and $y=1$ once about the y-axis, find the volume V of this solid.

[Sol] Since $x=e^y$,

$$V=\pi\int_{-1}^1 e^{2y}\,dy$$
$$=\pi\left[\frac{1}{2}e^{2y}\right]_{-1}^1$$
$$=\frac{\pi}{2}\left(e^2-\frac{1}{e^2}\right)$$



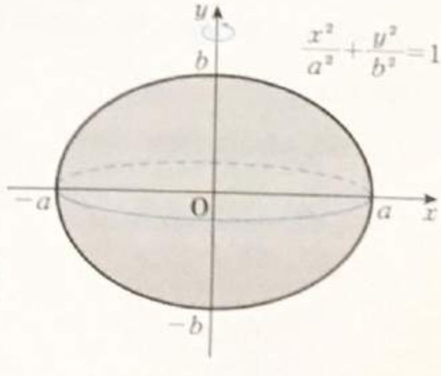
4. Let $a>0,\ b>0$. Given a solid formed by rotating the area enclosed by ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ once about the y-axis, find the volume V of this solid.

[Sol] Since $x^2=\frac{a^2}{b^2}(b^2-y^2)$,

$$V=\pi\int_{-b}^{b}\frac{a^2}{b^2}(b^2-y^2)\,dy$$
$$=\frac{2\pi a^2}{b^2}\int_0^b (b^2-y^2)\,dy$$

Even function

$$=\frac{2\pi a^2}{b^2}\left[b^2y-\frac{1}{3}y^3\right]_0^b$$
$$=\frac{4}{3}\pi a^2 b$$



Volumes

Name

Date / /

Time : to :

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Ex. Given a solid formed by rotating the area enclosed by curve $y=\sin x$ and line $y=\frac{1}{2}$ in $0 \leqq x \leqq \pi$ once about the x-axis, find the volume V of this solid.

[Sol] When $\sin x=\frac{1}{2}$, ⟵ Finding the x-coordinates of the points of intersection of curve $y=\sin x$ and line $y=\frac{1}{2}$

since $0 \leqq x \leqq \pi$, $x=\frac{\pi}{6}, \frac{5}{6}\pi$

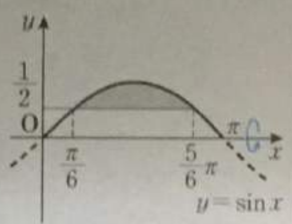
Since $\sin x \geqq \frac{1}{2}$ in $\frac{\pi}{6} \leqq x \leqq \frac{5}{6}\pi$,

$$V=\pi\int_{\frac{\pi}{6}}^{\frac{5}{6}\pi}\sin^2 x\,dx-\pi\int_{\frac{\pi}{6}}^{\frac{5}{6}\pi}\left(\frac{1}{2}\right)^2dx$$

$$=\pi\int_{\frac{\pi}{6}}^{\frac{5}{6}\pi}\left[\frac{1}{2}(1-\cos 2x)-\frac{1}{4}\right]dx$$

$$=\frac{\pi}{4}\int_{\frac{\pi}{6}}^{\frac{5}{6}\pi}(1-2\cos 2x)\,dx$$

$$=\frac{\pi}{4}\Big[x-\sin 2x\Big]_{\frac{\pi}{6}}^{\frac{5}{6}\pi}=\frac{\pi}{4}\left(\frac{2}{3}\pi+\sqrt{3}\right)$$



1. Given a solid formed by rotating the area enclosed by curve $y=\sin 2x$ and line $y=\frac{1}{\sqrt{2}}$ in $0 \leqq x \leqq \frac{\pi}{2}$ once about the x-axis, find the volume V of this solid.

[Sol] When $\sin 2x=\frac{1}{\sqrt{2}}$,

since $0 \leqq x \leqq \frac{\pi}{2}$, $x=\frac{\pi}{8}, \frac{3}{8}\pi$

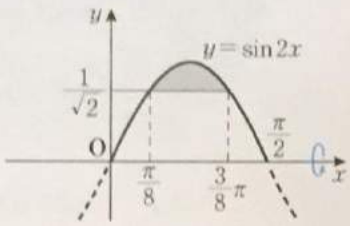
Since $\sin 2x \geqq \frac{1}{\sqrt{2}}$ in $\frac{\pi}{8} \leqq x \leqq \frac{3}{8}\pi$,

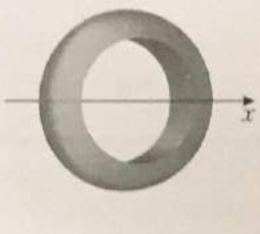
$$V=\pi\int_{\frac{\pi}{8}}^{\frac{3}{8}\pi}\sin^2 2x\,dx-\pi\int_{\frac{\pi}{8}}^{\frac{3}{8}\pi}\left(\frac{1}{\sqrt{2}}\right)^2dx$$

$$=\pi\int_{\frac{\pi}{8}}^{\frac{3}{8}\pi}\left[\frac{1}{2}(1-\cos 4x)-\frac{1}{2}\right]dx$$

$$=-\frac{\pi}{2}\int_{\frac{\pi}{8}}^{\frac{3}{8}\pi}\cos 4x\,dx$$

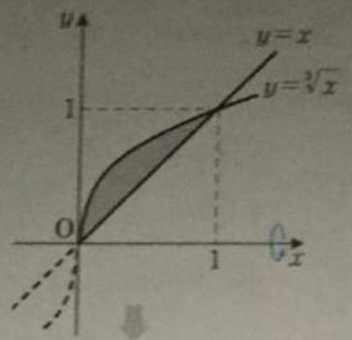
$$=-\frac{\pi}{2}\left[\frac{1}{4}\sin 4x\right]_{\frac{\pi}{8}}^{\frac{3}{8}\pi}=\frac{\pi}{4}$$





O145b

2. Given a solid formed by rotating the area enclosed by curve $y=\sqrt[3]{x}$ and line $y=x$ once about the x-axis, find the volume V of this solid. $(x \geqslant 0)$



[Sol] When $\sqrt[3]{x}=x$, $x^3-x=0$

$$x(x+1)(x-1)=0$$

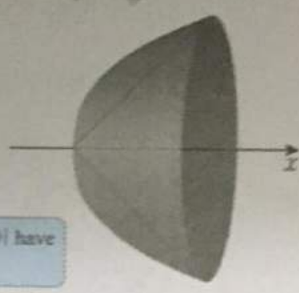
Since $x \geqslant 0$, $x=0,\ 1$

Since $\sqrt[3]{x} \geqslant x \geqslant 0$ in $0 \leqslant x \leqslant 1$, ← ※

$$V=\pi\int_0^1 (\sqrt[3]{x})^2\,dx-\pi\int_0^1 x^2\,dx$$

$$=\pi\int_0^1\left(x^{\frac{2}{3}}-x^2\right)dx$$

$$=\pi\left[\frac{3}{5}x^{\frac{5}{3}}-\frac{1}{3}x^3\right]_0^1=\frac{4}{15}\pi$$



※ To find the volume of revolution between $f(x)$ and $g(x)$, $|f(x)|$ and $|g(x)|$ have to be compared, i.e. $\sqrt[3]{x}$ and x are set greater than or equal to 0 in $0 \leqslant x \leqslant 1$.

3. Given a solid formed by rotating the area enclosed by two curves $y=\sin x$ and $y=\cos x$ and two lines $x=0$ and $x=\frac{\pi}{2}$ in $0 \leqslant x \leqslant \frac{\pi}{2}$ once about the x-axis, find the volume V of this solid.

[Sol] When $\sin x=\cos x$, $\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)=0$

Since $0 \leqslant x \leqslant \frac{\pi}{2}$, $x=\frac{\pi}{4}$

$a\sin\theta+b\cos\theta=\sqrt{a^2+b^2}\sin(\theta+\alpha)$

Since $\cos x \geqslant \sin x \geqslant 0$ in $0 \leqslant x \leqslant \frac{\pi}{4}$ and $\sin x \geqslant \cos x \geqslant 0$ in $\frac{\pi}{4} \leqslant x \leqslant \frac{\pi}{2}$,

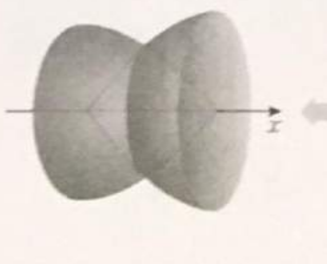
$$V=\left(\pi\int_0^{\frac{\pi}{4}}\cos^2 x\,dx-\pi\int_0^{\frac{\pi}{4}}\sin^2 x\,dx\right)+\left(\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin^2 x\,dx-\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2 x\,dx\right)$$

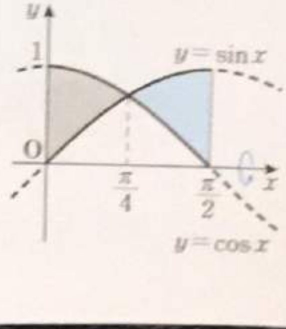
$$=\pi\int_0^{\frac{\pi}{4}}\cos 2x\,dx-\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos 2x\,dx$$

$\cos^2\alpha-\sin^2\alpha=\cos 2\alpha$

$$=\pi\left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}-\pi\left[\frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$=\pi$$





Volumes

Name

Date / /

Time : to :

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1. Let $0<r<b$. Given a solid formed by rotating circle $x^2+(y-b)^2=r^2$ once about the x-axis, find the volume V of this solid.

[Sol] Since $x^2+(y-b)^2=r^2$, $y=b\pm\sqrt{r^2-x^2}$

Since $(y-b)^2=r^2-x^2$, $y-b=\pm\sqrt{r^2-x^2}$

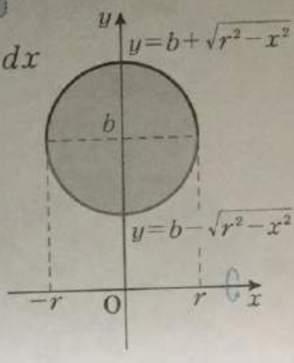
Therefore,

$$V=\pi\int_{-r}^{r}(b+\sqrt{r^2-x^2})^2\,dx-\pi\int_{-r}^{r}(b-\sqrt{r^2-x^2})^2\,dx$$

$$=4\pi b\int_{-r}^{r}\sqrt{r^2-x^2}\,dx$$

Even function

$$=8\pi b\int_{0}^{r}\sqrt{r^2-x^2}\,dx$$



Let $x=r\sin\theta$. $dx=r\cos\theta\,d\theta$ O105

Since $\cos\theta\geqq 0$ in $0\leqq\theta\leqq\frac{\pi}{2}$,

$$\sqrt{r^2-x^2}=\sqrt{r^2(1-\sin^2\theta)}=r\cos\theta$$

Thus,

x	$0\longrightarrow r$
θ	$0\longrightarrow\frac{\pi}{2}$

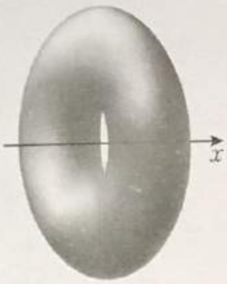
$$V=8\pi b\int_{0}^{\frac{\pi}{2}}r\cos\theta\cdot r\cos\theta\,d\theta$$

$$=8\pi r^2 b\int_{0}^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$$=4\pi r^2 b\int_{0}^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta$$

$$=4\pi r^2 b\left[\theta+\frac{1}{2}\sin 2\theta\right]_{0}^{\frac{\pi}{2}}$$

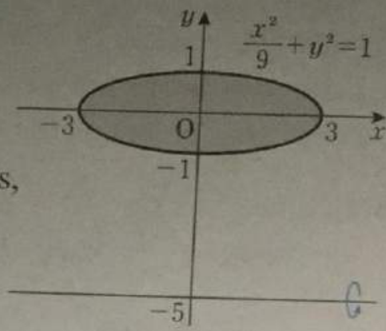
$$=2\pi^2 r^2 b$$



This type of revolution is called a ***torus***. The volume of the torus is equal to the product of the area of the circle πr^2 and the circumference $2\pi b$ given by the center of the circle $(0,\,b)$ rotating once about the x-axis.

0146b

2. Given a solid formed by rotating ellipse $\frac{x^2}{9}+y^2=1$ once about line $y=-5$, find the volume V of this solid.



[Sol] Since $\frac{x^2}{9}+y^2=1$, $y=\pm\frac{1}{3}\sqrt{9-x^2}$

Translating this ellipse 5 units along the y-axis,

$$y=\boxed{5\pm\frac{1}{3}\sqrt{9-x^2}}$$

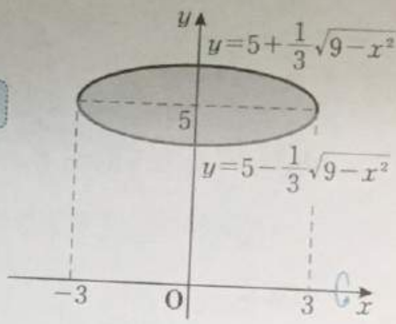
Therefore,

$$V=\pi\int_{-3}^{3}\left(5+\frac{1}{3}\sqrt{9-x^2}\right)^2dx-\pi\int_{-3}^{3}\left(5-\frac{1}{3}\sqrt{9-x^2}\right)^2dx$$

$$=\frac{20}{3}\pi\int_{-3}^{3}\sqrt{9-x^2}\,dx$$

$$=\frac{40}{3}\pi\int_{0}^{3}\sqrt{9-x^2}\,dx$$

Even function



Let $x=3\sin\theta$. $dx=3\cos\theta\,d\theta$

Since $\cos\theta\geqslant 0$ in $0\leqslant\theta\leqslant\frac{\pi}{2}$,

$$\sqrt{9-x^2}=\sqrt{9(1-\sin^2\theta)}=3\cos\theta$$

x	$0 \longrightarrow 3$
θ	$0 \longrightarrow \frac{\pi}{2}$

Thus,

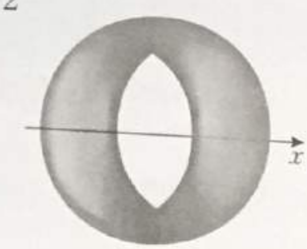
$$V=\frac{40}{3}\pi\int_{0}^{\frac{\pi}{2}}3\cos\theta\cdot3\cos\theta\,d\theta$$

$$=120\pi\int_{0}^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$$=60\pi\int_{0}^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta$$

$$=60\pi\left[\theta+\frac{1}{2}\sin 2\theta\right]_{0}^{\frac{\pi}{2}}$$

$$=30\pi^2$$



Volumes

Name

Date / /

Time : to :

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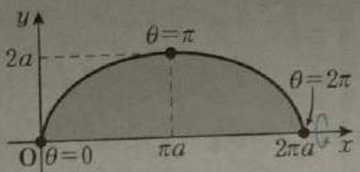
1. Let $a>0$. Given a solid formed by rotating the area enclosed by cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ $(0\leqq\theta\leqq 2\pi)$ and the x-axis once about the x-axis, find the volume V of this solid.



[Sol] $V=\pi\int_0^{2\pi a} y^2\,dx$, $dx=a(1-\cos\theta)\,d\theta$

x	$0 \longrightarrow 2\pi a$
θ	$0 \longrightarrow 2\pi$

Therefore,

$$V=\pi\int_0^{2\pi}[a(1-\cos\theta)]^2\cdot a(1-\cos\theta)\,d\theta$$
$$=\pi a^3\int_0^{2\pi}(1-\cos\theta)^3\,d\theta$$
$$=\pi a^3\int_0^{2\pi}(1-3\cos\theta+3\cos^2\theta-\cos^3\theta)\,d\theta$$

Then,

$$\int_0^{2\pi}d\theta=\Big[\theta\Big]_0^{2\pi}=2\pi$$
$$\int_0^{2\pi}\cos\theta\,d\theta=\Big[\sin\theta\Big]_0^{2\pi}=0$$
$$\int_0^{2\pi}\cos^2\theta\,d\theta=\frac{1}{2}\int_0^{2\pi}(1+\cos 2\theta)\,d\theta=\frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{2\pi}=\pi$$
$$\int_0^{2\pi}\cos^3\theta\,d\theta=\int_0^{2\pi}(1-\sin^2\theta)\cos\theta\,d\theta$$
$$=\int_0^{2\pi}(\cos\theta-\sin^2\theta\cos\theta)\,d\theta$$
$$=\left[\sin\theta-\frac{1}{3}\sin^3\theta\right]_0^{2\pi}$$
$$=0$$

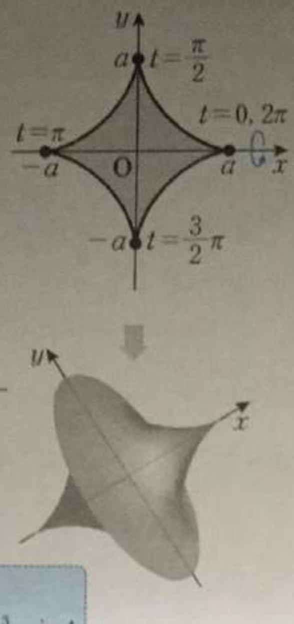
Thus,

$$V=\pi a^3(2\pi+3\pi)$$
$$=\mathbf{5\pi^2 a^3}$$

Since $\left(\frac{1}{3}\sin^3\theta\right)'=\sin^2\theta\cdot(\sin\theta)'=\sin^2\theta\cos\theta$, it can be $\int\sin^2\theta\cos\theta\,d\theta=\frac{1}{3}\sin^3\theta+C$.

0147b

2. Let $a>0$. Given a solid formed by rotating the area enclosed by curve $x=a\cos^3 t,\ y=a\sin^3 t\ (0\leqslant t\leqslant 2\pi)$ once about the x-axis, find the volume V of this solid.



[Sol] Since this curve is symmetric to both axes,

$$V=2\pi\int_0^a y^2\,dx$$

$$dx=3a\cos^2 t\cdot(-\sin t)dt=-3a\cos^2 t\sin t\,dt$$

Therefore,

x	$0\longrightarrow a$
t	$\frac{\pi}{2}\longrightarrow 0$

$$V=2\pi\int_{\frac{\pi}{2}}^{0}(a\sin^3 t)^2\cdot(-3a\cos^2 t\sin t)dt$$

$$=6\pi a^3\int_0^{\frac{\pi}{2}}\sin^7 t\cos^2 t\,dt$$

$$=6\pi a^3\int_0^{\frac{\pi}{2}}(1-\cos^2 t)^3\cos^2 t\sin t\,dt$$

$\sin^7 t=(\sin^2 t)^3\cdot\sin t$

$$=6\pi a^3\int_0^{\frac{\pi}{2}}(\cos^2 t-3\cos^4 t+3\cos^6 t-\cos^8 t)\sin t\,dt$$

$$=6\pi a^3\left[-\frac{1}{3}\cos^3 t+\frac{3}{5}\cos^5 t-\frac{3}{7}\cos^7 t+\frac{1}{9}\cos^9 t\right]_0^{\frac{\pi}{2}}$$ ※

$$=\frac{32}{105}\pi a^3$$

※ $\left(-\frac{1}{3}\cos^3 t\right)'=-\cos^2 t\cdot(\cos t)'=\cos^2 t\sin t$

Similarly, $\left(\frac{3}{5}\cos^5 t\right)'=-3\cos^4 t\sin t,\ \left(-\frac{3}{7}\cos^7 t\right)'=3\cos^6 t\sin t,\ \left(\frac{1}{9}\cos^9 t\right)'=-\cos^8 t\sin t$

$\therefore\ \int\cos^2 t\sin t\,dt=-\frac{1}{3}\cos^3 t+C,\ \int(-3\cos^4 t\sin t)dt=\frac{3}{5}\cos^5 t+C,$

$\int 3\cos^6 t\sin t\,dt=-\frac{3}{7}\cos^7 t+C,\ \int(-\cos^8 t\sin t)dt=\frac{1}{9}\cos^9 t+C$

Alternative Solution

$V=\pi\int_{-a}^{a}y^2\,dx,\ \ dx=3a\cos^2 t\cdot(-\sin t)dt=-3a\cos^2 t\sin t\,dt$

Therefore,

x	$-a\longrightarrow a$
t	$\pi\longrightarrow 0$

$$V=\pi\int_{\pi}^{0}(a\sin^3 t)^2\cdot(-3a\cos^2 t\sin t)dt$$

$$=3\pi a^3\int_0^{\pi}\sin^7 t\cos^2 t\,dt=3\pi a^3\int_0^{\pi}(1-\cos^2 t)^3\cos^2 t\sin t\,dt$$

$$=3\pi a^3\int_0^{\pi}(\cos^2 t-3\cos^4 t+3\cos^6 t-\cos^8 t)\sin t\,dt$$

$$=3\pi a^3\left[-\frac{1}{3}\cos^3 t+\frac{3}{5}\cos^5 t-\frac{3}{7}\cos^7 t+\frac{1}{9}\cos^9 t\right]_0^{\pi}=\frac{32}{105}\pi a^3$$

The curve shown in question 2 is called an ***astroid***.

Volumes

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
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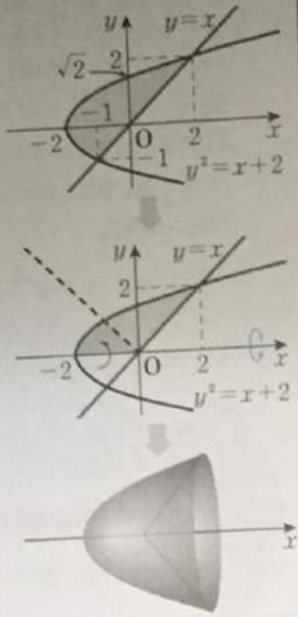
Ex. Given a solid formed by rotating the area enclosed by curve $y^2=x+2$ and line $y=x$ once about the x-axis, find the volume V of this solid.



[Sol] When $x+2=x^2$, ← Since $y=x$, $y^2=x^2$

$x^2-x-2=0$

$(x+1)(x-2)=0$

$\therefore\ x=-1,\ 2$

The x-coordinate of the point of intersection of $y^2=x+2$ and the x-axis is $x=\boxed{-2}$.

The x-coordinate of the point of intersection of $y=x$ and the x-axis is $x=\boxed{0}$.

Therefore,

$$V=\pi\int_{\boxed{-2}}^{2}(x+2)\,dx-\pi\int_{\boxed{0}}^{2}x^2\,dx$$

$$=\pi\left[\frac{1}{2}x^2+2x\right]_{\boxed{-2}}^{2}-\pi\left[\frac{1}{3}x^3\right]_{\boxed{0}}^{2}$$

$$=\boxed{\frac{16}{3}\pi}$$

Answers: -2, 0, -2, 0, -2, 0, $\frac{16}{3}\pi$

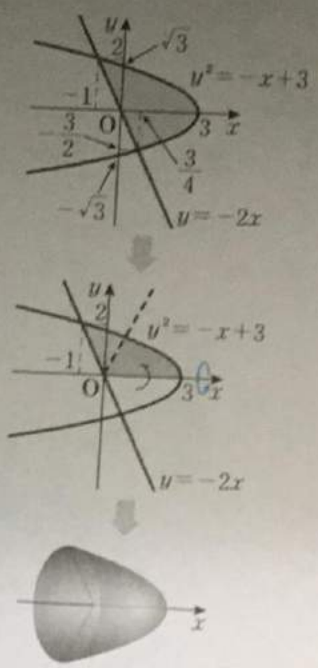
When the area enclosed by the curve exists on both sides of the rotation axis, some areas overlap when rotating. In such cases, it is possible to consider reflecting the enclosed area to the other side of the rotation axis.

O148b

1. Given a solid formed by rotating the area enclosed by curve $y^2=-x+3$ and line $y=-2x$ once about the x-axis, find the volume V of this solid.

[Sol] When $-x+3=4x^2$,

$$4x^2+x-3=0$$
$$(x+1)(4x-3)=0$$
$$\therefore\ x=-1,\ \frac{3}{4}$$



The x-coordinate of the point of intersection of $y^2=-x+3$ and the x-axis is $x=3$.
The x-coordinate of the point of intersection of $y=-2x$ and the x-axis is $x=0$.
Therefore,

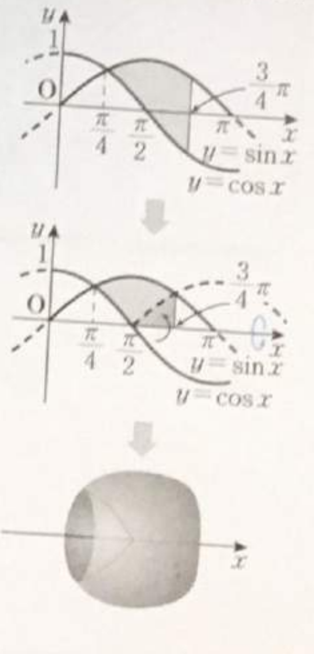
$$V=\pi\int_{-1}^{3}(-x+3)dx-\pi\int_{-1}^{0}4x^2dx$$
$$=\pi\left[-\frac{1}{2}x^2+3x\right]_{-1}^{3}-\pi\left[\frac{4}{3}x^3\right]_{-1}^{0}$$
$$=\frac{20}{3}\pi$$

2. Given a solid formed by rotating the area enclosed by two curves $y=\sin x$ and $y=\cos x$ and line $x=\frac{3}{4}\pi$ in $0\leqslant x\leqslant\pi$ once about the x-axis, find the volume V of this solid.

[Sol] When $\sin x=\cos x$,

$$\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)=0$$

$a\sin\theta+b\cos\theta$
$=\sqrt{a^2+b^2}\sin(\theta+\alpha)$



Since $0\leqslant x\leqslant\pi$, $x=\frac{\pi}{4}$
Since $0\leqslant x\leqslant\pi$, the x-coordinate of the point of intersection of $y=\cos x$ and the x-axis is $x=\frac{\pi}{2}$.
Therefore,

$$V=\pi\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi}\sin^2x\,dx-\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2x\,dx$$
$$=\frac{\pi}{2}\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi}(1-\cos 2x)dx-\frac{\pi}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1+\cos 2x)dx$$
$$=\frac{\pi}{2}\left[x-\frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{3}{4}\pi}-\frac{\pi}{2}\left[x+\frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$=\frac{\pi^2}{8}+\frac{3}{4}\pi\quad\left[=\frac{\pi}{8}(\pi+6)\right]$$

Volumes

Name

Date / /

Time : to :

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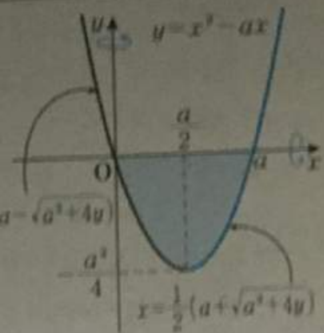
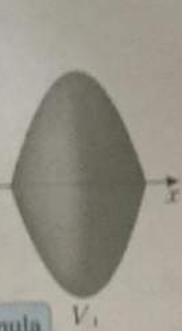
1. Let the volumes of the solid formed by rotating the area enclosed by curve $y=x^2-ax\ (a>0)$ and the x-axis once about the x-axis be V_1 and once about the y-axis be V_2 respectively. Determine the constant a such that $V_1=V_2$.



[Sol] Since $y=x^2-ax=x(x-a)=\left(x-\frac{a}{2}\right)^2-\frac{a^2}{4}$,

The points of intersection of the curve and the x-axis are $x=0, a$. The vertex is $\left(\frac{a}{2}, -\frac{a^2}{4}\right)$.

$$V_1=\pi\int_0^a (x^2-ax)^2\,dx$$
$$=\pi\int_0^a (x^4-2ax^3+a^2x^2)\,dx$$
$$=\pi\left[\frac{1}{5}x^5-\frac{1}{2}ax^4+\frac{1}{3}a^2x^3\right]_0^a$$
$$=\frac{1}{30}\pi a^5$$

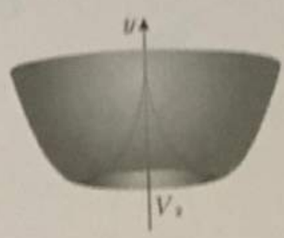


Also, since $y=x^2-ax$, $x^2-ax-y=0$

Quadratic Formula (J97)

Therefore, since $x=\frac{1}{2}(a\pm\sqrt{a^2+4y})$,

$$V_2=\pi\int_{-\frac{a^2}{4}}^{0}\left[\frac{1}{2}(a+\sqrt{a^2+4y})\right]^2dy-\pi\int_{-\frac{a^2}{4}}^{0}\left[\frac{1}{2}(a-\sqrt{a^2+4y})\right]^2dy$$
$$=\pi a\int_{-\frac{a^2}{4}}^{0}\sqrt{a^2+4y}\,dy$$
$$=\pi a\left[\frac{1}{6}(a^2+4y)^{\frac{3}{2}}\right]_{-\frac{a^2}{4}}^{0}$$
$$=\frac{1}{6}\pi a^4$$



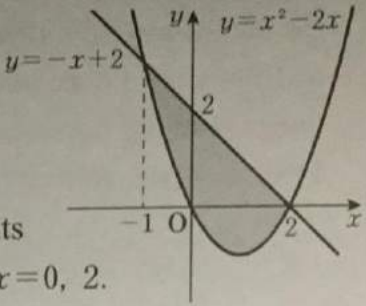
Since $V_1=V_2$, $\frac{1}{30}\pi a^5=\frac{1}{6}\pi a^4$

$a^4(a-5)=0$

Since $a>0$, $a=5$

0149b

2. Given a solid formed by rotating the area enclosed by parabola $y=x^2-2x$ and line $y=-x+2$ once about the x-axis, find the volume V of this solid.
(Pay attention to the diagram when reflecting the area with respect to the x-axis.)



[Sol] When $x^2-2x=-x+2$,

$x^2-x-2=0$

$(x+1)(x-2)=0$

$\therefore\ x=-1,\ 2$

Finding the x-coordinates of the points of intersection of $y=x^2-2x$ and $y=-x+2$

Since $x(x-2)=0$, the x-coordinates of the points of intersection of $y=x^2-2x$ and the x-axis are $x=0,\ 2$.
Also, the parabola given by reflecting $y=x^2-2x$ with respect to the x-axis is $y=-x^2+2x$; therefore,
when $-x^2+2x=-x+2$,

$x^2-3x+2=0$

$(x-1)(x-2)=0$

$\therefore\ x=1,\ 2$

Finding the x-coordinates of the points of intersection of $y=-x^2+2x$ and $y=-x+2$

Thus,

$$V=\left[\pi\int_{-1}^{0}(-x+2)^2\,dx-\pi\int_{-1}^{0}(x^2-2x)^2\,dx\right]+\pi\int_{0}^{1}(-x+2)^2\,dx$$

$$+\pi\int_{1}^{2}(-x^2+2x)^2\,dx$$

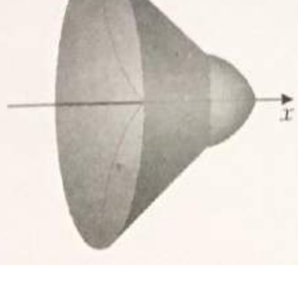
$$=\pi\int_{-1}^{0}(-x^4+4x^3-3x^2-4x+4)\,dx+\pi\int_{0}^{1}(x^2-4x+4)\,dx$$

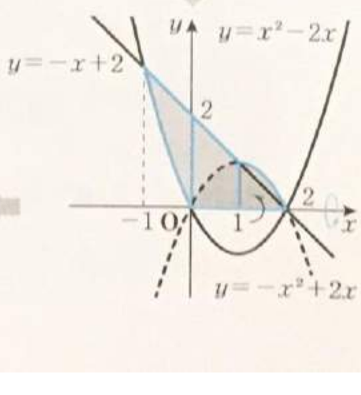
$$+\pi\int_{1}^{2}(x^4-4x^3+4x^2)\,dx$$

$$=\pi\left[-\frac{1}{5}x^5+x^4-x^3-2x^2+4x\right]_{-1}^{0}+\pi\left[\frac{1}{3}x^3-2x^2+4x\right]_{0}^{1}$$

$$+\pi\left[\frac{1}{5}x^5-x^4+\frac{4}{3}x^3\right]_{1}^{2}$$

$$=\frac{20}{3}\pi$$





Volumes

Name

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Time : to :

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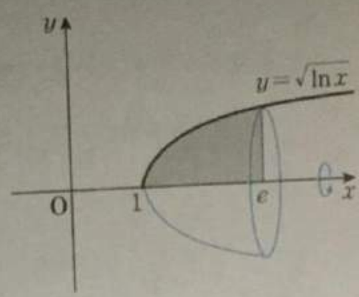
1. Given a solid formed by rotating the area enclosed by curve $y=\sqrt{\ln x}$, the x-axis and line $x=e$ once about the x-axis, find the volume V of this solid.

➡ O143



[Sol] $V=\pi\int_1^e(\sqrt{\ln x})^2\,dx$

$=\pi\int_1^e \ln x\,dx$

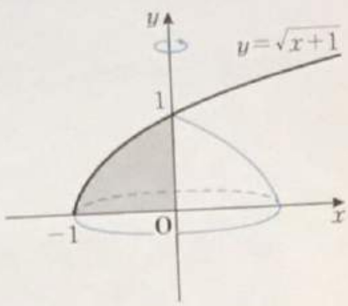
$=\pi\left(\left[x\ln x\right]_1^e-\int_1^e dx\right)$

$=\pi\left(e-\left[x\right]_1^e\right)$

$=\pi$

2. Given a solid formed by rotating the area enclosed by curve $y=\sqrt{x+1}$, the x-axis and the y-axis once about the y-axis, find the volume V of this solid.

➡ O144



[Sol] Since $x=y^2-1$,

$V=\pi\int_0^1(y^2-1)^2\,dy$

$=\pi\int_0^1(y^4-2y^2+1)\,dy$

$=\pi\left[\frac{1}{5}y^5-\frac{2}{3}y^3+y\right]_0^1$

$=\frac{8}{15}\pi$

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O150b

3. Given a solid formed by rotating the area enclosed by two curves $y=\sin x+1$ and $y=\cos x+1$ in $0 \leqslant x \leqslant 2\pi$ once about the x-axis, find the volume V of this solid.

➡ O145

[Sol] When $\sin x+1=\cos x+1$, $\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)=0$

Since $0 \leqslant x \leqslant 2\pi$, $x=\frac{\pi}{4}, \frac{5}{4}\pi$

Since $\sin x+1 \geqslant \cos x+1 \geqslant 0$ in $\frac{\pi}{4} \leqslant x \leqslant \frac{5}{4}\pi$,

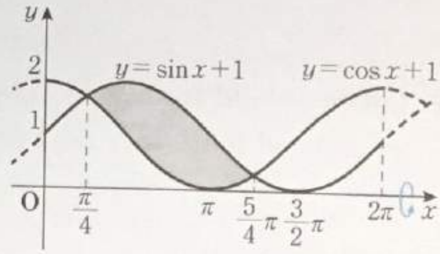
$$V=\pi\int_{\frac{\pi}{4}}^{\frac{5}{4}\pi}(\sin x+1)^2\,dx-\pi\int_{\frac{\pi}{4}}^{\frac{5}{4}\pi}(\cos x+1)^2\,dx$$

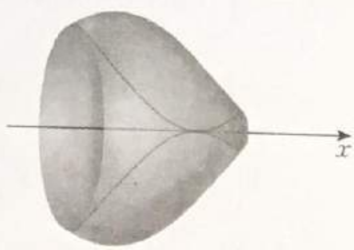
$$=\pi\int_{\frac{\pi}{4}}^{\frac{5}{4}\pi}(\sin^2 x-\cos^2 x+2\sin x-2\cos x)\,dx$$

$$=\pi\int_{\frac{\pi}{4}}^{\frac{5}{4}\pi}(-\cos 2x+2\sin x-2\cos x)\,dx$$

$$=\pi\left[-\frac{1}{2}\sin 2x-2\cos x-2\sin x\right]_{\frac{\pi}{4}}^{\frac{5}{4}\pi}$$

$$=4\sqrt{2}\pi$$





O151a

KUMON

Length of a Curve, Velocity and Distance

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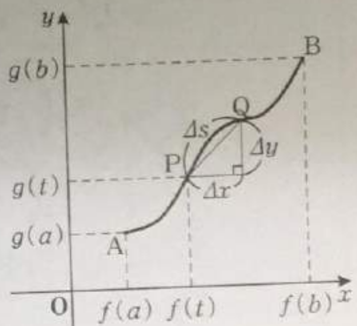
When the equations of a curve are expressed as

$$x=f(t),\ y=g(t) \quad (a \leqslant t \leqslant b)$$

using the parameter t, find the length L of the curve.

[Sol] Let $s(t)$ be the length of the curve from point $A(f(a), g(a))$ to point $P(f(t), g(t))$. Then, let the increments of x and y in terms of the increment of t, Δt, be Δx and Δy respectively.

When Δt is small enough, the increment of $s(t)$, Δs, is approximately equal to line segment PQ.



$$\therefore\ \Delta s \approx \sqrt{(\Delta x)^2+(\Delta y)^2}$$

$$\therefore\ \frac{\Delta s}{\Delta t} \approx \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2}$$

Then, let $\Delta t \to 0$. Since $\frac{\Delta x}{\Delta t} \to \frac{dx}{dt}$ and $\frac{\Delta y}{\Delta t} \to \frac{dy}{dt}$,

$$\frac{d}{dt}s(t)=\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}=\sqrt{\left(\boxed{\frac{dx}{dt}}\right)^2+\left(\boxed{\frac{dy}{dt}}\right)^2}$$

Therefore, $s(t)$ is an indefinite integral of the RHS of the above equation.

$$\int_a^b \sqrt{\left(\boxed{\frac{dx}{dt}}\right)^2+\left(\boxed{\frac{dy}{dt}}\right)^2}\,dt=s(b)-s(a)$$

Since $s(a)=0$ and $s(b)=L$,

When $t=a$, point P is located at the position of point A.
When $t=b$, point P is located at the position of point B.

$$L=\int_a^b \sqrt{\left(\boxed{\frac{dx}{dt}}\right)^2+\left(\boxed{\frac{dy}{dt}}\right)^2}\,dt$$

Answers: $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dx}{dt}$, $\frac{dy}{dt}$

0151b

Length of a Curve I

The length L of the curve $x=f(t)$, $y=g(t)$ $(a \leqslant t \leqslant b)$ is

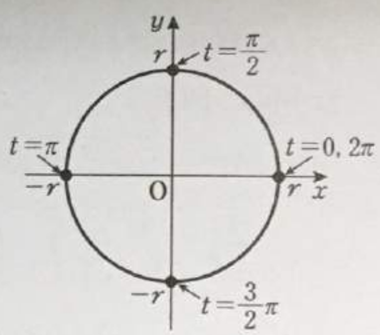
$$L=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_a^b \sqrt{[f'(t)]^2+[g'(t)]^2}\,dt$$

Find the length L of each given curve.

Ex. The circle $x=r\cos t$, $y=r\sin t$ $\quad (r>0,\ 0 \leqslant t \leqslant 2\pi)$

[Sol] $\dfrac{dx}{dt}=-r\sin t$, $\dfrac{dy}{dt}=r\cos t$

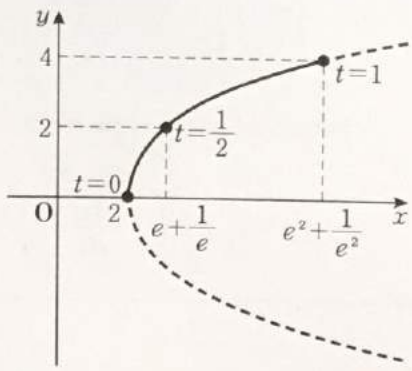
$$\begin{aligned}\therefore\ L&=\int_0^{2\pi}\sqrt{(-r\sin t)^2+(r\cos t)^2}\,dt\\&=\int_0^{2\pi}\sqrt{r^2}\,dt\\&=r\int_0^{2\pi}dt\\&=r\Big[t\Big]_0^{2\pi}\\&=2\pi r\end{aligned}$$



(1) $x=e^{2t}+e^{-2t}$, $y=4t$ $\quad (0 \leqslant t \leqslant 1)$

[Sol] $\dfrac{dx}{dt}=2e^{2t}-2e^{-2t}=2(e^{2t}-e^{-2t})$, $\dfrac{dy}{dt}=4$

$$\begin{aligned}\therefore\ L&=\int_0^1\sqrt{[2(e^{2t}-e^{-2t})]^2+4^2}\,dt\\&=\int_0^1\sqrt{4(e^{2t}+e^{-2t})^2}\,dt\\&=2\int_0^1(e^{2t}+e^{-2t})\,dt\\&=2\left[\frac{1}{2}e^{2t}-\frac{1}{2}e^{-2t}\right]_0^1\\&=e^2-\frac{1}{e^2}\end{aligned}$$



O152a

KUMON

Length of a Curve,
Velocity and Distance

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Find the length L of each given curve.

(1) $x=e^t\cos t,\ y=e^t\sin t \quad \left(0 \leqq t \leqq \frac{\pi}{2}\right)$

[Sol] $\dfrac{dx}{dt}=e^t\cos t+e^t\cdot(-\sin t)=e^t(\cos t-\sin t)$

$\dfrac{dy}{dt}=e^t\sin t+e^t\cos t=e^t(\sin t+\cos t)$

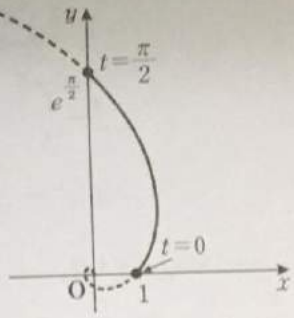
$$\therefore\ L=\int_0^{\frac{\pi}{2}}\sqrt{[e^t(\cos t-\sin t)]^2+[e^t(\sin t+\cos t)]^2}\,dt$$

$$=\int_0^{\frac{\pi}{2}}\sqrt{2e^{2t}}\,dt$$

$$=\sqrt{2}\int_0^{\frac{\pi}{2}}e^t\,dt$$

$$=\sqrt{2}\Big[e^t\Big]_0^{\frac{\pi}{2}}$$

$$=\sqrt{2}\left(e^{\frac{\pi}{2}}-1\right)$$



(2) The cycloid $x=t-\sin t,\ y=1-\cos t \quad (0 \leqq t \leqq 2\pi)$

[Sol] $\dfrac{dx}{dt}=1-\cos t,\ \dfrac{dy}{dt}=\sin t$

$$\therefore\ L=\int_0^{2\pi}\sqrt{(1-\cos t)^2+\sin^2 t}\,dt$$

$$=\int_0^{2\pi}\sqrt{2(1-\cos t)}\,dt$$

$\cos 2\alpha=1-2\sin^2\alpha$

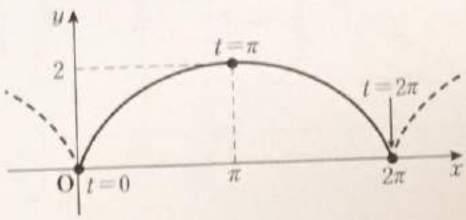
$$=\int_0^{2\pi}\sqrt{2\cdot 2\sin^2\frac{t}{2}}\,dt$$

When $0 \leqq t \leqq 2\pi,\ \sin\frac{t}{2} \geqq 0$

$$=2\int_0^{2\pi}\sin\frac{t}{2}\,dt$$

$$=2\left[-2\cos\frac{t}{2}\right]_0^{2\pi}$$

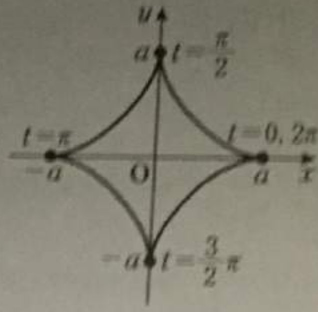
$$=8$$



O152b

(3) The astroid $x=a\cos^3 t,\ y=a\sin^3 t \quad (a>0,\ 0\leqq t\leqq 2\pi)$

[Sol] Since this curve is symmetric to both axes, L is 4 times as long as the length of the curve in $0\leqq t\leqq \frac{\pi}{2}$.



$$\frac{dx}{dt}=3a\cos^2 t\cdot(-\sin t)=-3a\cos^2 t\sin t$$

$$\frac{dy}{dt}=3a\sin^2 t\cos t$$

$$\therefore\ L=4\int_0^{\frac{\pi}{2}}\sqrt{(-3a\cos^2 t\sin t)^2+(3a\sin^2 t\cos t)^2}\,dt$$

$$=4\int_0^{\frac{\pi}{2}}\sqrt{9a^2\sin^2 t\cos^2 t}\,dt$$

$$=12a\int_0^{\frac{\pi}{2}}\sin t\cos t\,dt$$

When $0\leqq t\leqq\frac{\pi}{2}$, $\sin t\cos t\geqq 0$

$$=6a\int_0^{\frac{\pi}{2}}\sin 2t\,dt$$

$$=6a\left[-\frac{1}{2}\cos 2t\right]_0^{\frac{\pi}{2}}$$

$$=6a$$

Alternative Solution

$$\frac{dx}{dt}=3a\cos^2 t\cdot(-\sin t)=-3a\cos^2 t\sin t,\quad \frac{dy}{dt}=3a\sin^2 t\cos t$$

$$L=\int_0^{2\pi}\sqrt{(-3a\cos^2 t\sin t)^2+(3a\sin^2 t\cos t)^2}\,dt$$

$$=\int_0^{2\pi}\sqrt{9a^2\sin^2 t\cos^2 t}\,dt=3a\int_0^{2\pi}|\sin t\cos t|\,dt=\frac{3}{2}a\int_0^{2\pi}|\sin 2t|\,dt$$

$\sin 2t\geqq 0$ when $0\leqq t\leqq\frac{\pi}{2},\ \pi\leqq t\leqq\frac{3}{2}\pi$

$\sin 2t\leqq 0$ when $\frac{\pi}{2}\leqq t\leqq\pi,\ \frac{3}{2}\pi\leqq t\leqq 2\pi$

Therefore,

$$L=\frac{3}{2}a\left[\int_0^{\frac{\pi}{2}}\sin 2t\,dt+\int_{\frac{\pi}{2}}^{\pi}(-\sin 2t)\,dt+\int_{\pi}^{\frac{3}{2}\pi}\sin 2t\,dt+\int_{\frac{3}{2}\pi}^{2\pi}(-\sin 2t)\,dt\right]$$

$$=\frac{3}{2}a\left(\left[-\frac{1}{2}\cos 2t\right]_0^{\frac{\pi}{2}}+\left[\frac{1}{2}\cos 2t\right]_{\frac{\pi}{2}}^{\pi}+\left[-\frac{1}{2}\cos 2t\right]_{\pi}^{\frac{3}{2}\pi}+\left[\frac{1}{2}\cos 2t\right]_{\frac{3}{2}\pi}^{2\pi}\right)$$

$$=6a$$

Length of a Curve, Velocity and Distance

Name

Date / /

Time : to :

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When the equation of a curve is given by $y=f(x)$ $(a \leqq x \leqq b)$, consider

$x=t,\ y=f(t) \quad (a \leqq t \leqq b)$

Then, since $\dfrac{dx}{dt}=1$ and $\dfrac{dy}{dt}=\dfrac{dy}{dx}\cdot\dfrac{dx}{dt}=\dfrac{dy}{dx}=f'(x)$, the length L of the curve is given as follows.

Length of a Curve II

The length L of the curve $y=f(x)$ $(a \leqq x \leqq b)$ is

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

Find the length L of each given curve.

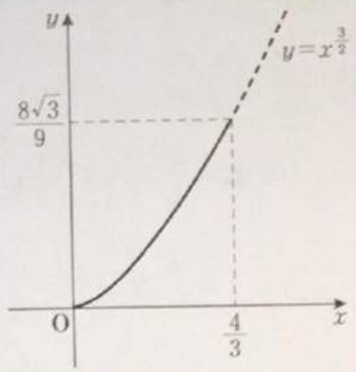
Ex. $y=x^{\frac{3}{2}}$ $\left(0 \leqq x \leqq \dfrac{4}{3}\right)$

[Sol] $\dfrac{dy}{dx}=\dfrac{3}{2}x^{\frac{1}{2}}$

$$\therefore\ L=\int_0^{\frac{4}{3}}\sqrt{1+\left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx$$

$$=\int_0^{\frac{4}{3}}\sqrt{1+\frac{9}{4}x}\,dx$$

$$=\left[\frac{8}{27}\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}\right]_0^{\frac{4}{3}}=\frac{56}{27}$$



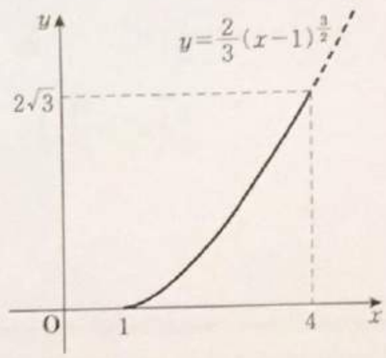
(1) $y=\dfrac{2}{3}(x-1)^{\frac{3}{2}}$ $\quad (1 \leqq x \leqq 4)$

[Sol] $\dfrac{dy}{dx}=(x-1)^{\frac{1}{2}}$

$$\therefore\ L=\int_1^4 \sqrt{1+\left[(x-1)^{\frac{1}{2}}\right]^2}\,dx$$

$$=\int_1^4 \sqrt{x}\,dx$$

$$=\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4=\frac{14}{3}$$



O153b

(2) $y=\frac{1}{3}x^3+\frac{1}{4x}$ $(1 \leqslant x \leqslant 3)$

[Sol] $\frac{dy}{dx}=x^2-\frac{1}{4x^2}$

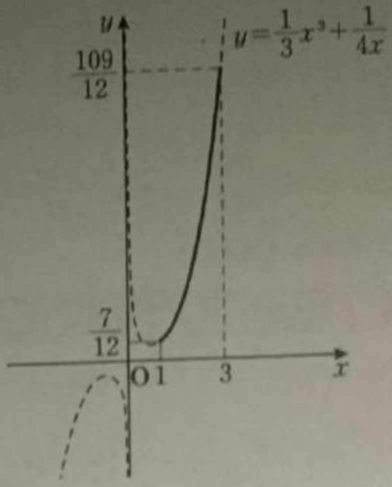
$$\therefore L=\int_1^3\sqrt{1+\left(x^2-\frac{1}{4x^2}\right)^2}\,dx$$

$$=\int_1^3\sqrt{\left(x^2+\frac{1}{4x^2}\right)^2}\,dx$$

$$=\int_1^3\left(x^2+\frac{1}{4x^2}\right)dx$$

$$=\left[\frac{1}{3}x^3-\frac{1}{4x}\right]_1^3$$

$$=\frac{53}{6}$$



(3) $y=\frac{1}{2}(e^x+e^{-x})$ $(0 \leqslant x \leqslant 2)$

[Sol] $\frac{dy}{dx}=\frac{1}{2}(e^x-e^{-x})$

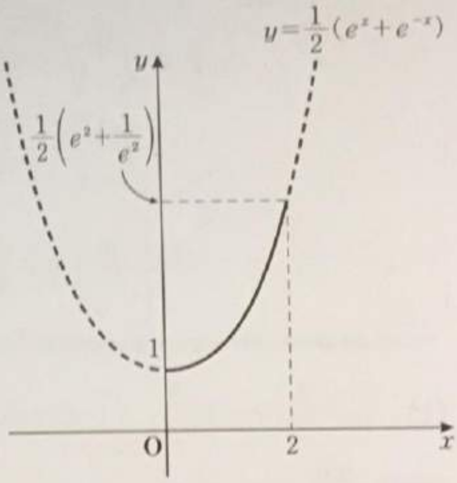
$$\therefore L=\int_0^2\sqrt{1+\left[\frac{1}{2}(e^x-e^{-x})\right]^2}\,dx$$

$$=\int_0^2\sqrt{\frac{1}{4}(e^x+e^{-x})^2}\,dx$$

$$=\frac{1}{2}\int_0^2(e^x+e^{-x})dx$$

$$=\frac{1}{2}\left[e^x-e^{-x}\right]_0^2$$

$$=\frac{1}{2}\left(e^2-\frac{1}{e^2}\right)$$



The curve shown in question (3) is called a ***catenary***. A catenary is the shape created when a flexible cord is hanging under its own weight.

O154a

KUMON

Length of a Curve, Velocity and Distance

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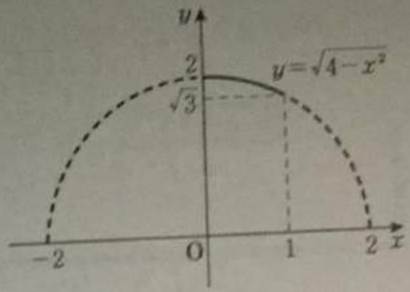
Find the length L of each given curve.

(1) $y=\sqrt{4-x^2}\quad(0\leqslant x\leqslant 1)$



[Sol] $\dfrac{dy}{dx}=-\dfrac{x}{\sqrt{4-x^2}}$

$$\therefore\ L=\int_0^1\sqrt{1+\left(-\frac{x}{\sqrt{4-x^2}}\right)^2}\,dx$$

$$=\int_0^1\sqrt{\frac{4}{4-x^2}}\,dx$$

$$=2\int_0^1\frac{dx}{\sqrt{4-x^2}}$$

Let $x=2\sin\theta$. $dx=2\cos\theta\,d\theta$

x	$0\longrightarrow 1$
θ	$0\longrightarrow\dfrac{\pi}{6}$

O105

When $0\leqslant\theta\leqslant\dfrac{\pi}{6}$, $\cos\theta>0$; therefore,

$$\sqrt{4-x^2}=\sqrt{4(1-\sin^2\theta)}=2\cos\theta$$

Therefore,

$$L=2\int_0^{\frac{\pi}{6}}\frac{1}{2\cos\theta}\cdot 2\cos\theta\,d\theta$$

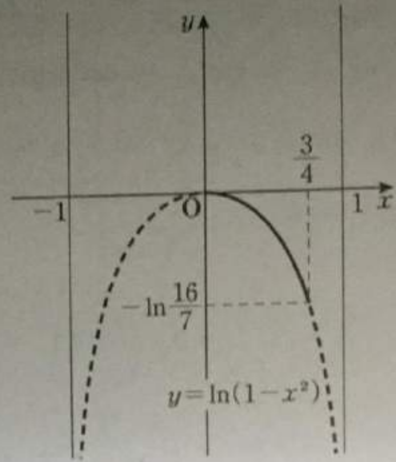
$$=2\int_0^{\frac{\pi}{6}}d\theta$$

$$=2\Big[\theta\Big]_0^{\frac{\pi}{6}}$$

$$=\frac{\pi}{3}$$

0154b

(2) $y=\ln(1-x^2)$ $\left(0 \leqslant x \leqslant \frac{3}{4}\right)$



[Sol] $\dfrac{dy}{dx}=-\dfrac{2x}{1-x^2}$

$$\therefore L=\int_0^{\frac{3}{4}}\sqrt{1+\left(-\frac{2x}{1-x^2}\right)^2}\,dx$$

$$=\int_0^{\frac{3}{4}}\sqrt{\left(\frac{1+x^2}{1-x^2}\right)^2}\,dx$$

$$=\int_0^{\frac{3}{4}}\frac{1+x^2}{1-x^2}\,dx$$

$$=\int_0^{\frac{3}{4}}\left(-1+\frac{2}{1-x^2}\right)dx$$

Let $\dfrac{2}{1-x^2}=\dfrac{2}{(1+x)(1-x)}=\dfrac{a}{1+x}+\dfrac{b}{1-x}$. ← 093

$$2=a(1-x)+b(1+x)$$

$$=(-a+b)x+(a+b)$$

$$\therefore \begin{cases}-a+b=0\\ a+b=2\end{cases}$$

$$\therefore a=1,\ b=1$$

Therefore,

$$L=\int_0^{\frac{3}{4}}\left(-1+\frac{1}{1+x}+\frac{1}{1-x}\right)dx$$

$$=\Big[-x+\ln|1+x|-\ln|1-x|\Big]_0^{\frac{3}{4}}$$

$$=\left[-x+\ln\left|\frac{1+x}{1-x}\right|\right]_0^{\frac{3}{4}}$$ ← $\ln M-\ln N=\ln\dfrac{M}{N}$

$$=-\frac{3}{4}+\ln 7$$

O155a

KUMON

Length of a Curve, Velocity and Distance

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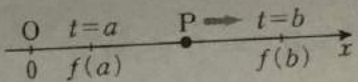
Let $x=f(t)$ be the coordinate of point P moving on a number line at time t, and let v be its velocity.



$$v=\frac{dx}{dt}=f'(t)$$ ← O46

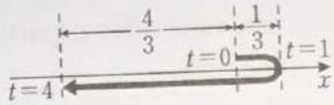
Therefore, the displacement s of point P from $t=a$ to $t=b$ and the distance l traveled by point P from $t=a$ to $t=b$ are given as follows.

Displacement and Distance of a Point Moving on a Line

$$s=\int_a^b v\,dt,\quad l=\int_a^b |v|dt$$

Ex. Given that the velocity v of point P moving on a number line at time t is $1-\sqrt{t}$, find the displacement s of point P and the distance l traveled by point P from $t=0$ to $t=4$.

[Sol] $s=\int_0^4(1-\sqrt{t})dt=\left[t-\frac{2}{3}t^{\frac{3}{2}}\right]_0^4=-\frac{4}{3}$



Also, $1-\sqrt{t}\geqslant 0$ when $0\leqslant t\leqslant 1$

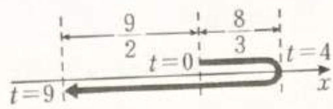
$1-\sqrt{t}\leqslant 0$ when $1\leqslant t\leqslant 4$

$$\therefore\ l=\int_0^1(1-\sqrt{t})dt+\int_1^4[-(1-\sqrt{t})]dt$$

$$=\left[t-\frac{2}{3}t^{\frac{3}{2}}\right]_0^1+\left[-t+\frac{2}{3}t^{\frac{3}{2}}\right]_1^4=2$$

1. Given that the velocity v of point P moving on a number line at time t is $2\sqrt{t}-t$, find the displacement s of point P and the distance l traveled by point P from $t=0$ to $t=9$.

[Sol] $s=\int_0^9(2\sqrt{t}-t)dt=\left[\frac{4}{3}t^{\frac{3}{2}}-\frac{1}{2}t^2\right]_0^9=-\frac{9}{2}$



Also, $2\sqrt{t}-t\geqslant 0$ when $0\leqslant t\leqslant 4$

$2\sqrt{t}-t\leqslant 0$ when $4\leqslant t\leqslant 9$

$$\therefore\ l=\int_0^4(2\sqrt{t}-t)dt+\int_4^9[-(2\sqrt{t}-t)]dt$$

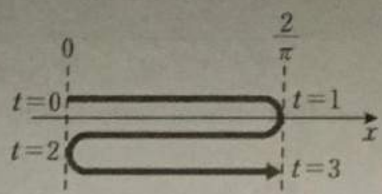
$$=\left[\frac{4}{3}t^{\frac{3}{2}}-\frac{1}{2}t^2\right]_0^4+\left[-\frac{4}{3}t^{\frac{3}{2}}+\frac{1}{2}t^2\right]_4^9=\frac{59}{6}$$

O155b

2. Two points P and Q moving on the x-axis start from the origin at the same time, and their velocities after t seconds are $\sin\pi t$ and $2\sin\pi t$ respectively.

(1) Find the coordinate s of point P at $t=3$.

[Sol] $s=\int_0^3 \sin\pi t\,dt=\left[-\frac{1}{\pi}\cos\pi t\right]_0^3=\frac{2}{\pi}$



(2) Find the distance l_1 traveled by point P from $t=0$ to $t=3$.

[Sol] $\sin\pi t \geqslant 0$ when $0\leqslant t\leqslant 1, 2\leqslant t\leqslant 3$

$\sin\pi t \leqslant 0$ when $1\leqslant t\leqslant 2$

$$\therefore\ l_1=\int_0^1 \sin\pi t\,dt+\int_1^2(-\sin\pi t)dt+\int_2^3 \sin\pi t\,dt$$

$$=\left[-\frac{1}{\pi}\cos\pi t\right]_0^1+\left[\frac{1}{\pi}\cos\pi t\right]_1^2+\left[-\frac{1}{\pi}\cos\pi t\right]_2^3=\frac{6}{\pi}$$

(3) Determine at what time (in seconds) two points P and Q meet again for the first time after departure. Then, find the distance l_2 traveled by point Q at this time.

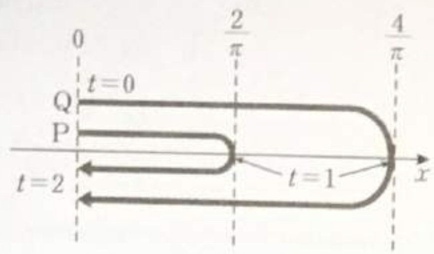
[Sol] If P and Q meet after t seconds,

$$\int_0^t \sin\pi t\,dt=\int_0^t 2\sin\pi t\,dt$$

$$\left[-\frac{1}{\pi}\cos\pi t\right]_0^t=\left[-\frac{2}{\pi}\cos\pi t\right]_0^t$$

$$-\frac{1}{\pi}(\cos\pi t-1)=-\frac{2}{\pi}(\cos\pi t-1)$$

$$\therefore\ \cos\pi t=1$$



Since $t>0$, minimum t is $\pi t=2\pi$, i.e. $t=2$; therefore, they meet **after 2 seconds**.

Also, $2\sin\pi t\geqslant 0$ when $0\leqslant t\leqslant 1$, and $2\sin\pi t\leqslant 0$ when $1\leqslant t\leqslant 2$

$$\therefore\ l_2=\int_0^1 2\sin\pi t\,dt+\int_1^2(-2\sin\pi t)dt$$

$$=\left[-\frac{2}{\pi}\cos\pi t\right]_0^1+\left[\frac{2}{\pi}\cos\pi t\right]_1^2=\frac{8}{\pi}$$

As shown below, if the function of t is integrated on an interval of integration which includes t, it stays as a function of t.

$$\int_0^t 2t\,dt=\left[t^2\right]_0^t=t^2$$

Length of a Curve, Velocity and Distance

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Change in the Velocity of a Point Moving on a Line

Let the velocity of point P moving on a number line at time t_0 and t_1 be v_0 and v_1 respectively, and let the acceleration at time t be α.

$$v_1 = v_0 + \int_{t_0}^{t_1} \alpha\, dt$$

Ex. Let the acceleration α of point P moving on the x-axis at time t be $-\pi^2 \sin \pi t$. When point P starts from the origin with an initial velocity (velocity at $t=0$) of π, find the velocity v and the coordinate s of point P after t seconds.

[Sol] $v = \pi + \int_0^t (-\pi^2 \sin \pi t)\, dt$

$= \pi + \left[\pi \cos \pi t\right]_0^t$

$= \pi \cos \pi t$

$s = \int_0^t \pi \cos \pi t\, dt$

$= \left[\sin \pi t\right]_0^t$

$= \sin \pi t$

1. Let the acceleration α of point P moving on the x-axis at time t be $\cos t + 2\cos 2t$. When point P starts from the origin with an initial velocity of 0, find the velocity v and the coordinate s of point P after t seconds.

[Sol] $v = 0 + \int_0^t (\cos t + 2\cos 2t)\, dt$

$= \left[\sin t + \sin 2t\right]_0^t$

$= \sin t + \sin 2t$

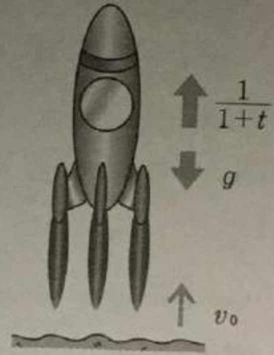
$s = \int_0^t (\sin t + \sin 2t)\, dt$

$= \left[-\cos t - \frac{1}{2}\cos 2t\right]_0^t$

$= -\cos t - \frac{1}{2}\cos 2t + \frac{3}{2}$

O156b

2. Let the acceleration due to gravity be g and the acceleration of a rocket after t seconds be $\frac{1}{1+t}$. It is launched straight upward from the ground with an initial velocity of v_0.



(1) Find the velocity v of the rocket and the height y reached by the rocket after t seconds.

[Sol] $v=v_0+\int_0^t\left(\frac{1}{1+t}-g\right)dt$

$=v_0+\Big[\ln|1+t|-gt\Big]_0^t$

$=\mathbf{v_0+\ln(1+t)-gt}$

Since $t>0$, $\ln|1+t|=\ln(1+t)$

$y=\int_0^t[v_0+\ln(1+t)-gt]dt$

$\int_0^t(v_0-gt)dt+\int_0^t\ln(1+t)dt$

$=\left[v_0t-\frac{1}{2}gt^2\right]_0^t+\Big[t\ln(1+t)\Big]_0^t-\int_0^t\frac{t}{1+t}dt$

$\int\ln(1+t)dt$
$=\int(t)'\ln(1+t)dt$
$=t\ln(1+t)-\int t[\ln(1+t)]'dt$
$=t\ln(1+t)-\int t\cdot\frac{1}{1+t}dt$

$=v_0t+t\ln(1+t)-\frac{1}{2}gt^2-\int_0^t\left(1-\frac{1}{1+t}\right)dt$

$=v_0t+t\ln(1+t)-\frac{1}{2}gt^2-\Big[t-\ln(1+t)\Big]_0^t$

$=\mathbf{(v_0-1)t+(1+t)\ln(1+t)-\frac{1}{2}gt^2}$

(2) Given that the rocket reaches its highest point after t_1 seconds, find the height h in terms of g and t_1. (Consider v at its highest point.)

[Sol] Since $v=v_0+\ln(1+t_1)-gt_1=0$,

When reaching the highest point, $v=0$

$v_0=gt_1-\ln(1+t_1)$

$\therefore\ h=(v_0-1)t_1+(1+t_1)\ln(1+t_1)-\frac{1}{2}gt_1^2$

$=[gt_1-\ln(1+t_1)-1]t_1+(1+t_1)\ln(1+t_1)-\frac{1}{2}gt_1^2$

$=\mathbf{\ln(1+t_1)+\frac{1}{2}gt_1^2-t_1}$

An object falls to the ground when it is released in the air.
The force which causes the object to fall towards the ground is called ***gravity***.

Length of a Curve, Velocity and Distance

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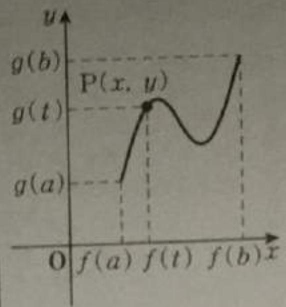
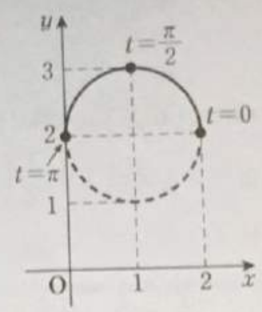
Let the coordinates of point $\mathrm{P}(x, y)$ moving on the plane at time t be $x=f(t)$ and $y=g(t)$. The distance l traveled by point P from time $t=a$ to $t=b$ can be considered the same as the length of a curve.
Therefore, the following equation is true.



Distance of a Point Moving on a Plane

$$l=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Ex. Given that the coordinates of point $\mathrm{P}(x, y)$ moving on the plane at time t are $x=\cos t+1$ and $y=\sin t+2$, find the distance l traveled by point P from $t=0$ to $t=\pi$.



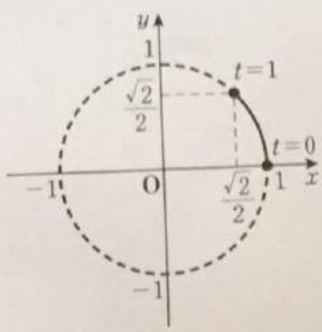
[Sol] $\frac{dx}{dt}=-\sin t,\ \frac{dy}{dt}=\cos t$

$$\therefore\ l=\int_0^{\pi}\sqrt{(-\sin t)^2+\cos^2 t}\,dt$$
$$=\int_0^{\pi}dt$$
$$=\Big[t\Big]_0^{\pi}=\pi$$

1. Given that the coordinates of point $\mathrm{P}(x, y)$ moving on the plane at time t are $x=\cos\frac{\pi}{4}t$ and $y=\sin\frac{\pi}{4}t$, find the distance l traveled by point P from $t=0$ to $t=1$.

[Sol] $\frac{dx}{dt}=-\frac{\pi}{4}\sin\frac{\pi}{4}t,\ \frac{dy}{dt}=\frac{\pi}{4}\cos\frac{\pi}{4}t$

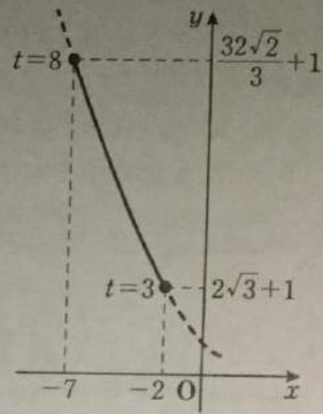
$$\therefore\ l=\int_0^1\sqrt{\left(-\frac{\pi}{4}\sin\frac{\pi}{4}t\right)^2+\left(\frac{\pi}{4}\cos\frac{\pi}{4}t\right)^2}\,dt$$
$$=\int_0^1\sqrt{\frac{\pi^2}{16}}\,dt$$
$$=\frac{\pi}{4}\int_0^1 dt$$
$$=\frac{\pi}{4}\Big[t\Big]_0^1=\frac{\pi}{4}$$



O157b

2. Given that the coordinates of point $P(x, y)$ moving on the plane at time t are $x=-t+1$ and $y=\frac{2}{3}t\sqrt{t}+1$, find the distance l traveled by point P from $t=3$ to $t=8$.

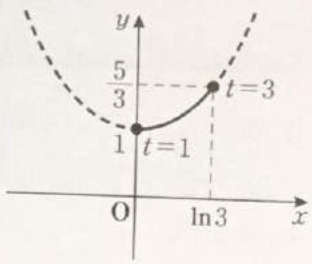
[Sol] $\frac{dx}{dt}=-1,\ \frac{dy}{dt}=\sqrt{t}$



$$\therefore\ l=\int_3^8 \sqrt{(-1)^2+(\sqrt{t})^2}\,dt$$
$$=\int_3^8 \sqrt{1+t}\,dt$$
$$=\left[\frac{2}{3}(1+t)^{\frac{3}{2}}\right]_3^8$$
$$=\frac{38}{3}$$

3. Given that the coordinates of point $P(x, y)$ moving on the plane at time t are $x=\ln t$ and $y=\frac{1}{2}\left(t+\frac{1}{t}\right)$, find the distance l traveled by point P from $t=1$ to $t=3$.

[Sol] $\frac{dx}{dt}=\frac{1}{t},\ \frac{dy}{dt}=\frac{1}{2}\left(1-\frac{1}{t^2}\right)$



$$\therefore\ l=\int_1^3 \sqrt{\left(\frac{1}{t}\right)^2+\left[\frac{1}{2}\left(1-\frac{1}{t^2}\right)\right]^2}\,dt$$
$$=\int_1^3 \sqrt{\frac{1}{4}\left(1+\frac{1}{t^2}\right)^2}\,dt$$
$$=\frac{1}{2}\int_1^3\left(1+\frac{1}{t^2}\right)dt$$
$$=\frac{1}{2}\left[t-\frac{1}{t}\right]_1^3$$
$$=\frac{4}{3}$$

O158a

KUMON

Length of a Curve, Velocity and Distance

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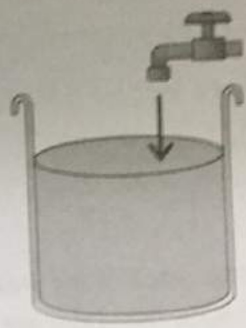
As with the displacement and distance, the change of volume can also be determined using integration.

Ex. An empty water tank can hold 200 ℓ of water. Given that the rate at which water is poured into the tank after t seconds is $\left(1+\frac{t}{50}\right)$ ℓ/sec, find how much time (in seconds) it takes to fill the tank.

[Sol] Let the amount of water after t seconds be $V(t)$.

$$V(t)=\int_0^t\left(1+\frac{t}{50}\right)dt$$
$$=\left[t+\frac{t^2}{100}\right]_0^t$$
$$=t+\frac{t^2}{100}$$
$$\therefore\ t+\frac{t^2}{100}=200$$
$$t^2+100t-20000=0$$
$$(t+200)(t-100)=0$$

Since $t>0$, $t=100$



Ans. 100 seconds

1. An empty bathtub can hold 270 ℓ of water. Given that the rate at which water is poured into the bathtub after t minutes is $15\sqrt{t}$ ℓ/min, determine how many more minutes it takes to fill the bathtub.

[Sol] Let the amount of water after t minutes be $V(t)$.

$$V(t)=\int_0^t 15\sqrt{t}\,dt$$
$$=\left[10t^{\frac{3}{2}}\right]_0^t$$
$$=10t^{\frac{3}{2}}$$
$$\therefore\ 10t^{\frac{3}{2}}=270$$
$$t^{\frac{3}{2}}=9^{\frac{3}{2}}$$
$$\therefore\ t=9$$

$\boxed{27=9^{\frac{3}{2}}}$

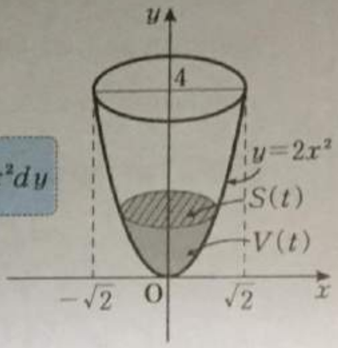
Ans. 9 minutes

O158b

2. Water is poured into the container formed by rotating curve $y=2x^2$ $(0 \leqq y \leqq 4)$ once about the y-axis at a rate of a per second. (a is a constant unrelated to the time.)

(1) Find how much time (in seconds) it takes to fill $\frac{1}{4}$ of the container.

[Sol] Let the volume of the container be V and the amount of water after t seconds be $V(t)$.



Since $x^2=\frac{1}{2}y$,

$$V=\pi\int_0^4 \frac{1}{2}y\,dy=\frac{\pi}{2}\left[\frac{1}{2}y^2\right]_0^4=4\pi \longleftarrow \boxed{V=\pi\int_c^d x^2 dy}$$

$$V(t)=at \quad \cdots①$$

Since $V(t)=\frac{1}{4}V$, $at=\frac{1}{4}\cdot 4\pi$

$\therefore\ t=\frac{\pi}{a}$ **Ans.** $\frac{\pi}{a}$ **seconds**

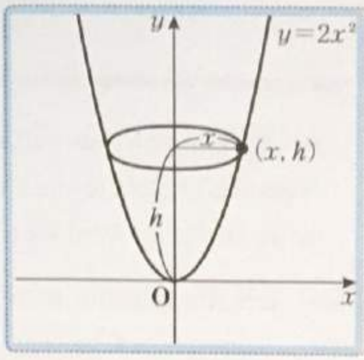
(2) Let the area of the water surface after t seconds be $S(t)$. Find the rate $S'(t)$ at which the water surface spreads.

[Sol] Let the depth of water after t seconds be h.

$$V(t)=\pi\int_0^h \frac{1}{2}y\,dy=\frac{\pi}{2}\left[\frac{1}{2}y^2\right]_0^h=\frac{1}{4}\pi h^2$$

From ①, $at=\frac{1}{4}\pi h^2 \quad \therefore\ h=2\sqrt{\frac{at}{\pi}}$

Let the radius of the water surface be x.



Since $h=2x^2$, $x^2=\frac{h}{2}$

$\therefore\ S(t)=\pi x^2=\pi\cdot\frac{h}{2}=\sqrt{\pi at}$

$\therefore\ S'(t)=\frac{1}{2}\sqrt{\frac{\pi a}{t}}$ **Ans.** $\frac{1}{2}\sqrt{\frac{\pi a}{t}}$ **per second**

(3) Find the rate at which the water surface spreads in the condition of (1).

[Sol] From (1) and (2), $S'\left(\frac{\pi}{a}\right)=\frac{a}{2}$ **Ans.** $\frac{a}{2}$ **per second**

Length of a Curve, Velocity and Distance

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1. Find the equation of curve C, $y=f(x)$ $(x \geqq 0)$, which satisfies the following conditions [1] and [2].
   [1] It passes through point $(0, 1)$.
   [2] The length L of the curve from point $(0, 1)$ to any arbitrary point (x, y) on curve C is given by $L=e^{2x}+y-2$.

[Sol] From [2], $\int_0^x \sqrt{1+[f'(t)]^2}\,dt=e^{2x}+f(x)-2$ ⟵ $L=\int_a^b \sqrt{1+[f'(t)]^2}\,dt$

Differentiating both sides with respect to x,

$$\sqrt{1+[f'(x)]^2}=2e^{2x}+f'(x)$$ ⟵ $\frac{d}{dx}\int_a^x f(t)dt=f(x)$

Squaring both sides,

$$1+[f'(x)]^2=4e^{4x}+4e^{2x}f'(x)+[f'(x)]^2$$

$$\therefore\ f'(x)=-e^{2x}+\frac{1}{4}e^{-2x}$$

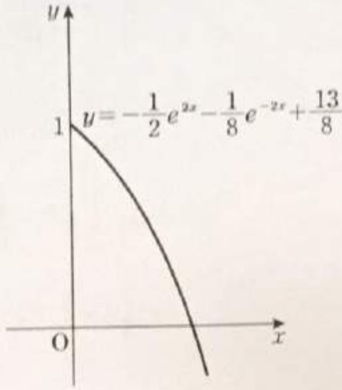
$$\therefore\ f(x)=\int\left(-e^{2x}+\frac{1}{4}e^{-2x}\right)dx=-\frac{1}{2}e^{2x}-\frac{1}{8}e^{-2x}+A$$

A is the constant of integration.

From [1], $f(0)=1$; therefore, $-\frac{1}{2}-\frac{1}{8}+A=1$

$$\therefore\ A=\frac{13}{8}$$

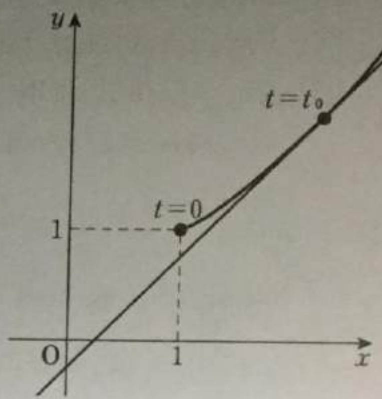
Thus, the equation is $y=-\frac{1}{2}e^{2x}-\frac{1}{8}e^{-2x}+\frac{13}{8}$.



O159b

2. Let the coordinates of point $P(x, y)$ moving on the xy-plane at time t be $x=1+\frac{5}{4}t^2$ and $y=1+t^{\frac{5}{2}}$ $(t \geqq 0)$.

(1) Find the time t_0 when the slope of the tangent to the curve formed by point P is 1.



[Sol] $\frac{dx}{dt}=\frac{5}{2}t$, $\frac{dy}{dt}=\frac{5}{2}t^{\frac{3}{2}}$

$$\therefore \frac{dy}{dx}=\frac{\frac{5}{2}t^{\frac{3}{2}}}{\frac{5}{2}t}=t^{\frac{1}{2}} \quad \Leftarrow \quad \boxed{\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

Therefore, since $t_0^{\frac{1}{2}}=1$, $\mathbf{t_0=1}$

(2) Find the distance traveled by point P from $t=0$ to $t=t_0$.

[Sol] Let the distance traveled by point P be l. From (1),

$$l=\int_0^1 \sqrt{\left(\frac{5}{2}t\right)^2+\left(\frac{5}{2}t^{\frac{3}{2}}\right)^2}\,dt \quad \Leftarrow \quad \boxed{t_0=1}$$

$$=\int_0^1 \sqrt{\frac{25}{4}t^2(1+t)}\,dt$$

$$=\frac{5}{2}\int_0^1 t\sqrt{1+t}\,dt$$

Let $\sqrt{1+t}=u$. Since $1+t=u^2$, $t=u^2-1$, $dt=2u\,du$

$$\therefore l=\frac{5}{2}\int_1^{\sqrt{2}}(u^2-1)u\cdot 2u\,du$$

t	$0 \longrightarrow 1$
u	$1 \longrightarrow \sqrt{2}$

$$=5\int_1^{\sqrt{2}}(u^4-u^2)\,du$$

$$=5\left[\frac{1}{5}u^5-\frac{1}{3}u^3\right]_1^{\sqrt{2}}$$

$$=\mathbf{\frac{2}{3}(\sqrt{2}+1)}$$

Length of a Curve, Velocity and Distance

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Name

Date / /

Time : to :

1. Find the length L of the following curve. ➡ O151

$$x=t\cos t-\sin t,\ y=t\sin t+\cos t \quad (0\leqq t\leqq 2\pi)$$

[Sol] $\dfrac{dx}{dt}=[1\cdot\cos t+t\cdot(-\sin t)]-\cos t=-t\sin t$

$\dfrac{dy}{dt}=(1\cdot\sin t+t\cos t)-\sin t=t\cos t$

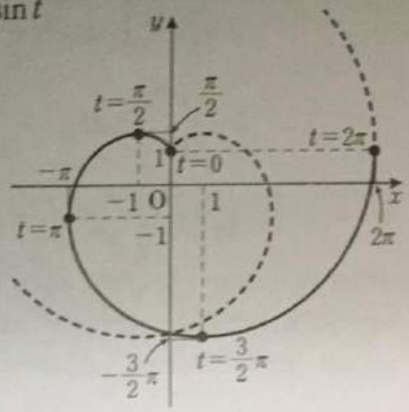
$$\therefore\ L=\int_0^{2\pi}\sqrt{(-t\sin t)^2+(t\cos t)^2}\,dt$$

$$=\int_0^{2\pi}\sqrt{t^2}\,dt$$

$$=\int_0^{2\pi}t\,dt$$

$$=\left[\frac{1}{2}t^2\right]_0^{2\pi}$$

$$=2\pi^2$$



2. Find the length L of curve $y=\sqrt{x}-\dfrac{x\sqrt{x}}{3}$ $(0\leqq x\leqq 4)$. ➡ O153

[Sol] $\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{x}}-\dfrac{\sqrt{x}}{2}=\dfrac{1}{2}\left(\dfrac{1}{\sqrt{x}}-\sqrt{x}\right)$

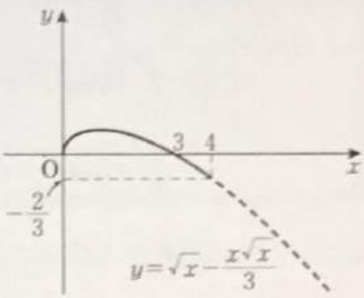
$$\therefore\ L=\int_0^4\sqrt{1+\left[\frac{1}{2}\left(\frac{1}{\sqrt{x}}-\sqrt{x}\right)\right]^2}\,dx$$

$$=\int_0^4\sqrt{\frac{1}{4}\left(\frac{1}{\sqrt{x}}+\sqrt{x}\right)^2}\,dx$$

$$=\frac{1}{2}\int_0^4\left(\frac{1}{\sqrt{x}}+\sqrt{x}\right)dx$$

$$=\frac{1}{2}\left[2x^{\frac{1}{2}}+\frac{2}{3}x^{\frac{3}{2}}\right]_0^4$$

$$=\frac{14}{3}$$



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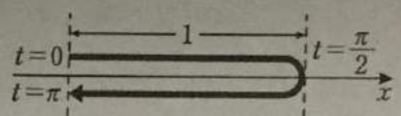
O160b

3. Given that the velocity v of point P moving on a number line at time t is $\sin 2t$, find the displacement s of point P and the distance l traveled by point P from $t=0$ to $t=\pi$.

➡ O155

[Sol] $s=\int_0^{\pi} \sin 2t\,dt=\left[-\frac{1}{2}\cos 2t\right]_0^{\pi}=0$



Also, $\sin 2t \geqslant 0$ when $0 \leqslant t \leqslant \frac{\pi}{2}$

$\sin 2t \leqslant 0$ when $\frac{\pi}{2} \leqslant t \leqslant \pi$

$$\therefore\ l=\int_0^{\frac{\pi}{2}} \sin 2t\,dt+\int_{\frac{\pi}{2}}^{\pi}(-\sin 2t)\,dt$$

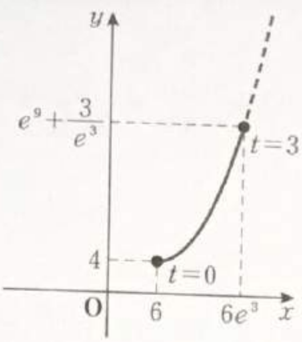
$$=\left[-\frac{1}{2}\cos 2t\right]_0^{\frac{\pi}{2}}+\left[\frac{1}{2}\cos 2t\right]_{\frac{\pi}{2}}^{\pi}$$

$$=2$$

4. Given that the coordinates of point $\mathrm{P}(x, y)$ moving on the plane at time t are $x=6e^t$ and $y=e^{3t}+3e^{-t}$, find the distance l traveled by point P from $t=0$ to $t=3$.

➡ O157

[Sol] $\frac{dx}{dt}=6e^t,\ \frac{dy}{dt}=3e^{3t}-3e^{-t}=3(e^{3t}-e^{-t})$



$$\therefore\ l=\int_0^3 \sqrt{(6e^t)^2+[3(e^{3t}-e^{-t})]^2}\,dt$$

$$=\int_0^3 \sqrt{9(e^{3t}+e^{-t})^2}\,dt$$

$$=3\int_0^3 (e^{3t}+e^{-t})\,dt$$

$$=3\left[\frac{1}{3}e^{3t}-e^{-t}\right]_0^3$$

$$=e^9-\frac{3}{e^3}+2$$

Differential Equations

Name

Date / /

Time : to :

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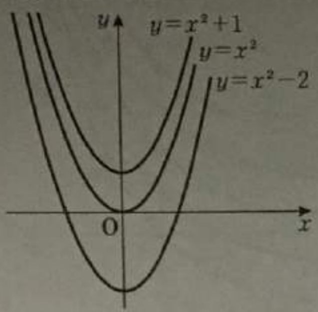


Express the slope of the tangent to each of the following curves at point (x, y) in terms of x.

[1] $y=x^2+1$ ··· $2x$ ← $y'=2x$

[2] $y=x^2$ ··· $\boxed{2x}$

[3] $y=x^2-2$ ··· $\boxed{2x}$

Answers: All the answers are the same, $2x$

From [1]~[3] above, if the slope of the tangent to the curve $y=f(x)$ at point (x, y) is always equal to $2x$, then there exists an infinite number of different curves in the form of $y=x^2+C$, where C is an arbitrary constant. This can be determined as follows.

Ex. Given that the slope of the tangent to the curve $y=f(x)$ at point $P(x, y)$ is always equal to $2x$, find the equation of the curve.

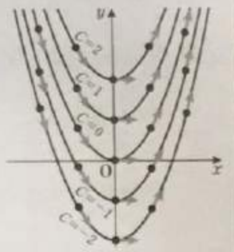
[Sol] Since the slope of the tangent at point P is $\frac{dy}{dx}$,

$$\frac{dy}{dx}=2x$$

Integrating both sides with respect to x,

$$y=\int 2x\,dx=x^2+C$$

(C is an arbitrary constant)



1. Given that the slope of the tangent to the curve $y=f(x)$ at point $P(x, y)$ is always equal to x^2, find the equation of the curve.

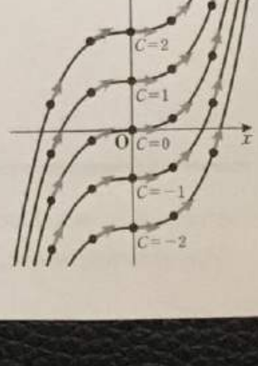
[Sol] Since the slope of the tangent at point P is $\frac{dy}{dx}$,

$$\frac{dy}{dx}=x^2$$

Integrating both sides with respect to x,

$$y=\int x^2\,dx=\frac{1}{3}x^3+C$$

(C is an arbitrary constant)



O161b

2. Given that the slope of the tangent to the curve $y=f(x)$ at point $P(x, y)$ is always equal to $e^{\frac{x}{2}}$, find the equation of the curve.

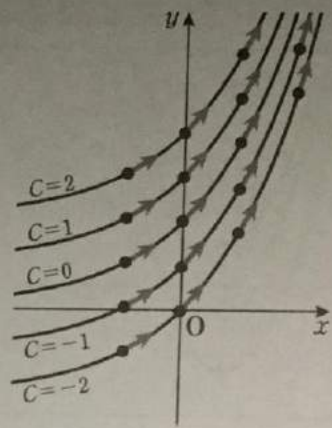
[Sol] Since the slope of the tangent at point P is $\frac{dy}{dx}$,

$$\frac{dy}{dx}=e^{\frac{x}{2}}$$

Integrating both sides with respect to x,

$$y=\int e^{\frac{x}{2}}dx=2e^{\frac{x}{2}}+C$$

(C is an arbitrary constant)



3. Given that the slope of the tangent to the curve $y=f(x)$ at point $P(x, y)$ is always equal to $\frac{1}{x}$, find the equation of the curve. $(x>0)$

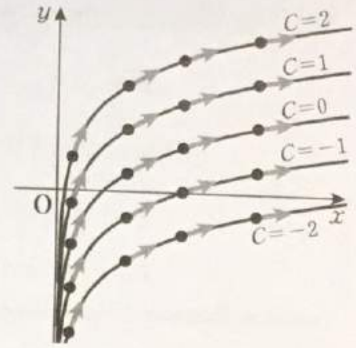
[Sol] Since the slope of the tangent at point P is $\frac{dy}{dx}$,

$$\frac{dy}{dx}=\frac{1}{x}$$

Integrating both sides with respect to x,

$$y=\int \frac{1}{x}dx=\ln x+C \quad \longleftarrow \quad \boxed{\text{Since } x>0,\ \ln|x|=\ln x}$$

(C is an arbitrary constant)



An equation which contains derivatives of an unknown function, such as $\frac{dy}{dx}=2x$, is called a ***differential equation***.

Differential Equations

Name

Date / /

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Time : to :

1. For each given equation, find the differential equation by eliminating the arbitrary constant C.

Ex. $y=Cx+1$ ···①

[Sol] Differentiating both sides with respect to x,

$$\frac{dy}{dx}=C \quad \cdots②$$

Substituting ② into ①, $y=x\frac{dy}{dx}+1$

(1) $y=C(x+1)$ ···①

[Sol] Differentiating both sides with respect to x,

$$\frac{dy}{dx}=C \quad \cdots②$$

Substituting ② into ①, $\boldsymbol{y=(x+1)\frac{dy}{dx}}$

(2) $y^2=4Cx$ ···①

[Sol] Differentiating both sides with respect to x,

$$2y\frac{dy}{dx}=4C \quad \cdots②$$

Substituting ② into ①, $y^2=2xy\frac{dy}{dx}$

$\therefore\ \boldsymbol{y=2x\frac{dy}{dx}}$

Assuming $y\neq 0$ when dividing both sides by y; however, $y=2x\frac{dy}{dx}$ is also true for $y=0$.

(3) $x^2+y^2=C^2$

[Sol] Differentiating both sides with respect to x,

$$2x+2y\frac{dy}{dx}=0$$

$\therefore\ \boldsymbol{x+y\frac{dy}{dx}=0}$

O162b

2. For each given equation, find the differential equation by eliminating the arbitrary constants A and B.

(1) $y=Ax^2+B$

[Sol] Differentiating both sides with respect to x,

$$\frac{dy}{dx}=\boxed{2Ax}\quad\cdots①$$

Differentiating both sides again with respect to x,

$$\frac{d^2y}{dx^2}=\boxed{2A}\quad\cdots②$$

Substituting ② into ①, $\frac{dy}{dx}=\boxed{x\frac{d^2y}{dx^2}}$

(2) $y=\frac{1}{2}A+\frac{3}{4}Bx^2$

[Sol] Differentiating both sides with respect to x,

$$\frac{dy}{dx}=\frac{3}{2}Bx\quad\cdots①$$

Differentiating both sides again with respect to x,

$$\frac{d^2y}{dx^2}=\frac{3}{2}B\quad\cdots②$$

Substituting ② into ①, $\frac{dy}{dx}=x\frac{d^2y}{dx^2}$

(3) $y=Ae^{kx}+Be^{-kx}\quad\cdots①$

[Sol] Differentiating both sides with respect to x,

$$\frac{dy}{dx}=Ake^{kx}-Bke^{-kx}$$

Differentiating both sides again with respect to x,

$$\frac{d^2y}{dx^2}=Ak^2e^{kx}+Bk^2e^{-kx}=k^2(Ae^{kx}+Be^{-kx})\quad\cdots②$$

From ① and ②, $\frac{d^2y}{dx^2}=k^2y$

Differential Equations

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
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Solve the following differential equations.

Ex. $\dfrac{dy}{dx}=x^2-x+2$

[Sol] Integrating both sides with respect to x,

$$y=\int(x^2-x+2)dx$$
$$=\frac{1}{3}x^3-\frac{1}{2}x^2+2x+C \quad (C \text{ is an arbitrary constant})$$

(1) $\dfrac{dy}{dx}=x^3-2x$

[Sol] Integrating both sides with respect to x,

$$y=\int(x^3-2x)dx$$
$$=\frac{1}{4}x^4-x^2+C \quad (C \text{ is an arbitrary constant})$$

(2) $\dfrac{dy}{dx}=\ln x+1$

[Sol] Integrating both sides with respect to x,

$$y=\int(\ln x+1)dx$$
$$=\left(x\ln x-\int dx\right)+x$$
$$=x\ln x+C \quad (C \text{ is an arbitrary constant})$$

$$\int \ln x\,dx=\int(x)'\ln x\,dx$$
$$=x\ln x-\int x(\ln x)'dx$$
$$=x\ln x-\int x\cdot\frac{1}{x}dx$$

A function which satisfies a given differential equation is called the ***solution*** to the differential equation.

0163b

(3) $\dfrac{d^2y}{dx^2}=\dfrac{1}{x^3}$

[Sol] Integrating both sides with respect to x,

$$\frac{dy}{dx}=\int\frac{1}{x^3}dx$$

$$=-\frac{1}{2x^2}+C_1$$

Integrating both sides again with respect to x,

$$y=\int\left(-\frac{1}{2x^2}+C_1\right)dx$$

$$=\frac{1}{2x}+C_1x+C_2 \quad \textbf{(}C_1 \textbf{ and } C_2 \textbf{ are arbitrary constants)}$$

(4) $\dfrac{d^2y}{dx^2}=xe^x$

[Sol] Integrating both sides with respect to x,

$$\frac{dy}{dx}=\int xe^x dx$$

$$=xe^x-\int e^x dx$$

$$=xe^x-e^x+C_1$$

$$\int xe^x dx=\int x(e^x)'dx$$
$$=xe^x-\int (x)'e^x dx$$

Integrating both sides again with respect to x,

$$y=\int(xe^x-e^x+C_1)dx$$

$$=\left(xe^x-\int e^x dx\right)-e^x+C_1x$$

$$=(x-2)e^x+C_1x+C_2 \quad \textbf{(}C_1 \textbf{ and } C_2 \textbf{ are arbitrary constants)}$$

Differential Equations

Name

Date / /

Time : to :

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Solve the following differential equations.

Ex. $\dfrac{dy}{dx}=\dfrac{x}{y}$

[Sol] $y\dfrac{dy}{dx}=x$ ← Rearranging into the form $f(y)\dfrac{dy}{dx}=g(x)$

Integrating both sides with respect to x,

$\int y\,dy=\int x\,dx$ ← LHS $=\int y\dfrac{dy}{dx}\cdot dx=\int y\,dy$

$\therefore\ \dfrac{1}{2}y^2=\dfrac{1}{2}x^2+C_1$

$\therefore\ y^2=x^2+2C_1$

Generally, arbitrary constants do not have coefficients.

Let $2C_1=C$.

$y^2=x^2+C$ (C is an arbitrary constant)

(1) $\dfrac{dy}{dx}=\dfrac{3x}{y}$

[Sol] $y\dfrac{dy}{dx}=3x$

Integrating both sides with respect to x,

$\int y\,dy=\int 3x\,dx$

$\therefore\ \dfrac{1}{2}y^2=\dfrac{3}{2}x^2+C_1$

$\therefore\ y^2=3x^2+2C_1$

Let $2C_1=C$.

$y^2=3x^2+C$ (**C is an arbitrary constant**)

As shown in **Ex.**, the differential equation expressed as $f(y)\dfrac{dy}{dx}=g(x)$ is called a ***separable differential equation***.

O164b

(2) $\dfrac{dy}{dx}=\dfrac{2}{y}$

[Sol] $y\dfrac{dy}{dx}=2$

Integrating both sides with respect to x,

$$\int y\,dy=\int 2dx$$

$$\therefore\ \frac{1}{2}y^2=2x+C_1$$

$$\therefore\ y^2=4x+2C_1$$

Let $2C_1=C$.

$y^2=4x+C$ **(C is an arbitrary constant)**

(3) $\dfrac{dy}{dx}=\dfrac{3}{\sqrt{y}}$

[Sol] $\sqrt{y}\dfrac{dy}{dx}=3$

Integrating both sides with respect to x,

$$\int \sqrt{y}\,dy=\int 3dx$$

$$\therefore\ \frac{2}{3}y\sqrt{y}=3x+C_1$$

$$\therefore\ y\sqrt{y}=\frac{9}{2}x+\frac{3}{2}C_1$$

Let $\frac{3}{2}C_1=C$.

$y\sqrt{y}=\dfrac{9}{2}x+C$ **(C is an arbitrary constant)**

[$2y\sqrt{y}=9x+C$ **(C is an arbitrary constant)**]

O165a

KUMON

Differential Equations

Name

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Time : to :

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Solve the following differential equations.

Ex. $\frac{dy}{dx} = -y \quad \cdots ①$

Since $\frac{dy}{dx}=0$ when $y=0$, this satisfies ①.

[Sol] (i) It is obvious that constant function $y=0$ is a solution.

(ii) When $y \neq 0$, rearranging ①,

$$\frac{1}{y} \cdot \frac{dy}{dx} = -1$$

Rearranging into the form $f(y)\frac{dy}{dx} = g(x)$

Integrating both sides with respect to x,

$$\int \frac{dy}{y} = \int (-1)\,dx$$

LHS $= \int \frac{1}{y} \cdot \frac{dy}{dx} \cdot dx = \int \frac{dy}{y}$

$\therefore \ \ln|y| = -x + C_1$

$C = \pm e^{C_1} \neq 0$

$\therefore \ y = \pm e^{-x+C_1} = \pm e^{C_1} \cdot e^{-x}$

Let $\pm e^{C_1} = C$. $y = Ce^{-x}$ (C is an arbitrary constant, $C \neq 0$)

When $C=0$ in (ii), $y=0$, which is the same as (i).

$\therefore \ y = Ce^{-x}$ (C is an arbitrary constant)

Solutions can be compiled.

(1) $\frac{dy}{dx} = 4xy \quad \cdots ①$

[Sol] (i) It is obvious that constant function $y=0$ is a solution.

(ii) When $y \neq 0$, rearranging ①,

$$\frac{1}{y} \cdot \frac{dy}{dx} = 4x$$

Integrating both sides with respect to x,

$$\int \frac{dy}{y} = \int 4x\,dx$$

$\therefore \ \ln|y| = 2x^2 + C_1$

$\therefore \ y = \pm e^{2x^2+C_1} = \pm e^{C_1} \cdot e^{2x^2}$

Let $\pm e^{C_1} = C$. $y = Ce^{2x^2}$ (C is an arbitrary constant, $C \neq 0$)

When $C=0$ in (ii), $y=0$, which is the same as (i).

$\therefore \ y = Ce^{2x^2}$ **(C is an arbitrary constant)**

A function which always gives a constant value, such as $y=3$ and $y=-5$, is called a ***constant function***.

0165b

(2) $\dfrac{dy}{dx}=y\cos x$ ⋯①

[Sol] (i) It is obvious that constant function $y=0$ is a solution.

(ii) When $y\neq0$, rearranging ①,

$$\frac{1}{y}\cdot\frac{dy}{dx}=\cos x$$

Integrating both sides with respect to x,

$$\int\frac{dy}{y}=\int\cos x\,dx$$

$\therefore\ \ln|y|=\sin x+C_1$

$\therefore\ y=\pm e^{\sin x+C_1}=\pm e^{C_1}\cdot e^{\sin x}$

Let $\pm e^{C_1}=C$. $y=Ce^{\sin x}$ (C is an arbitrary constant, $C\neq0$)

When $C=0$ in (ii), $y=0$, which is the same as (i).

$\therefore\ y=Ce^{\sin x}$ **(C is an arbitrary constant)**

(3) $y=x\dfrac{dy}{dx}+1$ ⋯①

[Sol] Rearranging ①, $x\dfrac{dy}{dx}=y-1$ ⋯②

(i) It is obvious that constant function $y=1$ is a solution.

(ii) When $y\neq1$, $x\neq0$; therefore, rearranging ②,

When $y\neq1$, ② cannot be satisfied if $x=0$.

$$\frac{1}{y-1}\cdot\frac{dy}{dx}=\frac{1}{x}$$

Integrating both sides with respect to x,

$$\int\frac{dy}{y-1}=\int\frac{dx}{x}$$

$\ln M+\ln N=\ln MN$

$\therefore\ \ln|y-1|=\ln|x|+C_1=\ln|x|+\ln e^{C_1}=\ln e^{C_1}|x|$

$\therefore\ y-1=\pm e^{C_1}x$

When $|\alpha+\beta|=|\gamma|$, $\alpha+\beta=\pm\gamma$

Let $\pm e^{C_1}=C$. $y=Cx+1$ (C is an arbitrary constant, $C\neq0$)

When $C=0$ in (ii), $y=1$, which is the same as (i).

$\therefore\ y=Cx+1$ **(C is an arbitrary constant)**

O166a

KUMON

Differential Equations

Name

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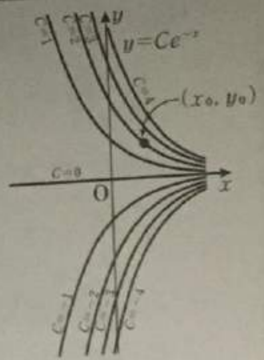


Among the solutions of the differential equations, the one which contains an arbitrary constant C such as $y=Ce^{-x}$ in **Ex.** on O 165 is called the ***general solution***.

The group of the curves on the right can be obtained by substituting various values into the constant C of $y=Ce^{-x}$. Such curves obtained by the solutions of the differential equations are called the ***integral curves***. Among these integral curves, there is only one curve passing through the given point (x_0, y_0). This means there is only one solution that satisfies the following condition: When $x=x_0$, $y=y_0$ $\cdots$①

The condition such as ①, which determines the value of the arbitrary constant in the general solution, is called the ***initial condition***. When the arbitrary constant is determined by an initial condition, the solution is called the ***particular solution***.

Find the particular solutions to the following differential equations given the initial conditions in the brackets [ ].

Ex. $(x+3)\dfrac{dy}{dx}=y-2$ $\cdots$① [When $x=-1$, $y=1$]

[Sol] Constant functions $x=-3$ and $y=2$ do not satisfy the given condition.

When $x=-1$, $y=1$; therefore, constant functions $x=-3$ and $y=2$ are not the solutions.

When $x\neq-3$ and $y\neq2$, rearranging ①,

$$\frac{1}{y-2}\cdot\frac{dy}{dx}=\frac{1}{x+3}$$

Integrating both sides with respect to x,

$$\int\frac{dy}{y-2}=\int\frac{dx}{x+3}$$

$$\therefore\ \ln|y-2|=\ln|x+3|+C_1=\ln|x+3|+\ln e^{C_1}=\ln e^{C_1}|x+3|$$

$$\therefore\ y-2=\pm e^{C_1}(x+3)$$

Let $\pm e^{C_1}=C$. $y=C(x+3)+2$

When $x=-1$, $y=1$; therefore, $C=\boxed{-\dfrac{1}{2}}$

$$\therefore\ y=\boxed{-\frac{1}{2}}(x-1)$$

Answers: All the answers are the same, $-\dfrac{1}{2}$

O166b

(1) $(x-1)\dfrac{dy}{dx}=y \cdots ①$ [When $x=0$, $y=1$]

[Sol] Constant functions $x=1$ and $y=0$ do not satisfy the given condition.

When $x\neq 1$ and $y\neq 0$, rearranging ①,

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{x-1}$$

Integrating both sides with respect to x,

$$\int\frac{dy}{y}=\int\frac{dx}{x-1}$$

$$\therefore\ \ln|y|=\ln|x-1|+C_1=\ln|x-1|+\ln e^{C_1}=\ln e^{C_1}|x-1|$$

$$\therefore\ y=\pm e^{C_1}(x-1)$$

Let $\pm e^{C_1}=C$. $y=C(x-1)$

When $x=0$, $y=1$; therefore, $C=-1$

$$\therefore\ \boldsymbol{y=-(x-1)}\quad \left[y=-x+1\right]$$

(2) $\sqrt{1+x}\dfrac{dy}{dx}=\sqrt{1+y} \cdots ①$ [When $x=3$, $y=8$]

[Sol] Constant functions $x=-1$ and $y=-1$ do not satisfy the given condition.

When $x\neq -1$ and $y\neq -1$, rearranging ①,

$$\frac{1}{\sqrt{1+y}}\cdot\frac{dy}{dx}=\frac{1}{\sqrt{1+x}}$$

Integrating both sides with respect to x,

$$\int\frac{dy}{\sqrt{1+y}}=\int\frac{dx}{\sqrt{1+x}}$$

$$\therefore\ 2\sqrt{1+y}=2\sqrt{1+x}+C$$

When $x=3$, $y=8$; therefore, $C=2$

$$\therefore\ \boldsymbol{\sqrt{1+y}=\sqrt{1+x}+1}$$

Differential Equations

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

Ex. Solve the following differential equation using $x+y=u$.

$$\frac{dy}{dx}=x+y \quad \cdots①$$

[Sol] Differentiating both sides of $x+y=u$ with respect to x,

$$1+\frac{dy}{dx}=\frac{du}{dx} \quad \cdots②$$

From ① and ②, $\dfrac{du}{dx}=u+1 \quad \cdots③$ ⟵

$$\frac{du}{dx}=1+\frac{dy}{dx}$$
$$=1+x+y$$
$$=u+1$$

(i) When $u=-1$,

$y=-x-1$ and this satisfies ①; therefore this is a solution. ⟵

Differentiating both sides of $y=-x-1$ with respect to x,

$$\frac{dy}{dx}=-1$$

Since $u=-1$, ① is satisfied.

(ii) When $u\neq-1$, rearranging ③,

$$\frac{1}{u+1}\cdot\frac{du}{dx}=1$$

Integrating both sides with respect to x,

$$\int\frac{du}{u+1}=\int dx$$

$\therefore\ \ln|u+1|=x+C_1$

$\therefore\ u+1=\pm e^{x+C_1}=\pm e^{C_1}\cdot e^x$

Let $\pm e^{C_1}=C$. $u=Ce^x-\boxed{1}$

(C is an arbitrary constant, $C\neq0$)

When $C=0$ in (ii), $u=-1$, which is the same as (i).

$\therefore\ u=Ce^x-\boxed{1}$

$\therefore\ y=Ce^x-x-\boxed{1}$ (C is an arbitrary constant)

Answers: All the answers are the same, 1

O167b

1. Solve the following differential equation using $3x+y=u$.

$$\frac{dy}{dx}+(3x+y)=0 \quad \cdots ①$$

[Sol] Differentiating both sides of $3x+y=u$ with respect to x,

$$3+\frac{dy}{dx}=\frac{du}{dx} \quad \cdots ②$$

From ① and ②, $\frac{du}{dx}=-(u-3) \quad \cdots ③$

(i) When $u=3$,

$y=-3x+3$ and this satisfies ①; therefore this is a solution.

(ii) When $u\neq3$, rearranging ③,

$$\frac{1}{u-3}\cdot\frac{du}{dx}=-1$$

Integrating both sides with respect to x,

$$\int\frac{du}{u-3}=\int(-1)dx$$

$\therefore\ \ln|u-3|=-x+C_1$

$\therefore\ u-3=\pm e^{-x+C_1}=\pm e^{C_1}\cdot e^{-x}$

Let $\pm e^{C_1}=C$. $u=Ce^{-x}+3$ (C is an arbitrary constant, $C\neq0$)

When $C=0$ in (ii), $u=3$, which is the same as (i).

$\therefore\ u=Ce^{-x}+3$

$\therefore\ y=Ce^{-x}-3x+3$ **(C is an arbitrary constant)**

C is an arbitrary constant; therefore, generally, it does not have coefficients.

Differential Equations

Name

Date / /

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

Time : to :

1. Solve the following differential equation using $x+y=u$.

$$\frac{dy}{dx}=\frac{1-x-y}{x+y} \quad \cdots①$$

[Sol] Differentiating both sides of $x+y=u$ with respect to x,

$$1+\frac{dy}{dx}=\frac{du}{dx} \quad \cdots②$$

From ① and ②, $\frac{du}{dx}=1+\frac{1-u}{u}=\frac{1}{u}$, i.e. $u\frac{du}{dx}=1$

Integrating both sides with respect to x,

$$\int u\,du=\int dx$$

$$\therefore\ \frac{1}{2}u^2=x+C_1$$

$$\therefore\ (x+y)^2=2x+2C_1$$

Let $2C_1=C$.

$(x+y)^2=2x+C$ **(C is an arbitrary constant)**

O168b

2. Find the particular solution to the following differential equation given the initial condition in the brackets [ ].

$x\dfrac{dy}{dx}-(y-x)=0 \cdots①$ [When $x=1$, $y=1$]

$\left(\text{Use } \dfrac{y}{x}=u \text{ when } x\neq 0\right)$

[Sol] Constant function $x=0$ does not satisfy the given condition.

When $x\neq 0$, rearranging ①,

$\dfrac{dy}{dx}=\dfrac{y}{x}-1 \cdots②$

Let $\dfrac{y}{x}=u$. $y=ux$

Differentiating both sides with respect to x,

$\dfrac{dy}{dx}=\dfrac{du}{dx}\cdot x+u\cdot 1$ ⟵ $\boxed{y'=u'x+ux'}$

$=x\dfrac{du}{dx}+u \cdots③$

From ② and ③, $x\dfrac{du}{dx}+u=u-1$, i.e. $\dfrac{du}{dx}=-\dfrac{1}{x}$

Integrating both sides with respect to x,

$\int du=\int\left(-\dfrac{1}{x}\right)dx$

$\therefore\ u=-\ln|x|+C$

$\therefore\ y=x(-\ln|x|+C)$

When $x=1$, $y=1$; therefore, $C=1$

$\therefore\ y=x(-\ln|x|+1)$

O169a

KUMON

Differential Equations

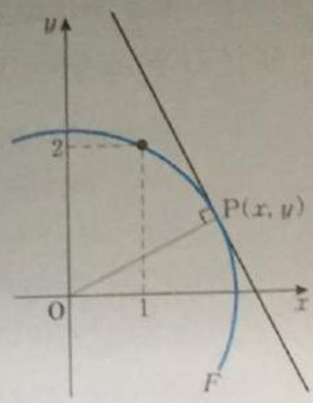
Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	1	2~

1. The tangent to curve F at point $P(x, y)$ is always perpendicular to the line connecting origin O and point P. Given that this curve passes through point $(1, 2)$, find the equation of curve F.



[Sol] The slope of the tangent at point P is $\dfrac{dy}{dx}$.

Since the slope of line OP is $\dfrac{y}{x}$,

$$\frac{dy}{dx}\cdot\frac{y}{x}=-1$$ ← Perpendicular Condition (M15)

So, $y\dfrac{dy}{dx}=-x$

Integrating both sides with respect to x,

$$\int y\,dy=\int(-x)\,dx$$

$$\therefore\ \frac{1}{2}y^2=-\frac{1}{2}x^2+C$$

$$\therefore\ x^2+y^2=2C$$

Since curve F passes through point $(1, 2)$,

$$1^2+2^2=2C$$

$$\therefore\ \boldsymbol{x^2+y^2=5}$$

0169b

2. The functions $f(x)$ and $g(x)$ satisfy the following equations.

$f'(x)+2g'(x)-f(x)-2g(x)=0 \quad \cdots①$

$2f'(x)-g'(x)+x[2f(x)-g(x)]=0 \quad \cdots②$

(1) Find $f(x)+2g(x)$.

[Sol] Let $y=f(x)+2g(x)$. From ①, $\dfrac{dy}{dx}-y=0 \quad \cdots③$

(i) It is obvious that constant function $y=0$ is a solution of ③.

(ii) When $y\neq0$, rearranging ③, $\dfrac{1}{y}\cdot\dfrac{dy}{dx}=1$

Integrating both sides with respect to x, $\displaystyle\int\frac{dy}{y}=\int dx$

$\therefore \ln|y|=x+C_1 \quad \therefore y=\pm e^{x+C_1}=\pm e^{C_1}\cdot e^x$

Let $\pm e^{C_1}=C_2$. $y=C_2e^x$ (C_2 is an arbitrary constant, $C_2\neq0$)

When $C_2=0$ in (ii), $y=0$, which is the same as (i).

$\therefore$ $\boldsymbol{f(x)+2g(x)=C_2e^x}$ **(C_2 is an arbitrary constant)**

(2) Find $2f(x)-g(x)$.

[Sol] Let $y=2f(x)-g(x)$. From ②, $\dfrac{dy}{dx}+xy=0 \quad \cdots④$

(i) It is obvious that constant function $y=0$ is a solution of ④.

(ii) When $y\neq0$, rearranging ④, $\dfrac{1}{y}\cdot\dfrac{dy}{dx}=-x$

Integrating both sides with respect to x, $\displaystyle\int\frac{dy}{y}=\int(-x)dx$

$\therefore \ln|y|=-\dfrac{1}{2}x^2+C_3 \quad \therefore y=\pm e^{-\frac{1}{2}x^2+C_3}=\pm e^{C_3}\cdot e^{-\frac{1}{2}x^2}$

Let $\pm e^{C_3}=C_4$. $y=C_4e^{-\frac{1}{2}x^2}$ (C_4 is an arbitrary constant, $C_4\neq0$)

When $C_4=0$ in (ii), $y=0$, which is the same as (i).

$\therefore$ $\boldsymbol{2f(x)-g(x)=C_4e^{-\frac{1}{2}x^2}}$ **(C_4 is an arbitrary constant)**

(3) Given $f(0)=5$ and $g(0)=0$, find $f(x)$ and $g(x)$.

[Sol] From (1) and (2), $f(x)+2g(x)=C_2e^x \quad \cdots⑤$,

$2f(x)-g(x)=C_4e^{-\frac{1}{2}x^2} \quad \cdots⑥$

From ⑤ and ⑥, $f(x)=\dfrac{1}{5}\left(C_2e^x+2C_4e^{-\frac{1}{2}x^2}\right)$,

$g(x)=\dfrac{1}{5}\left(2C_2e^x-C_4e^{-\frac{1}{2}x^2}\right)$

Since $f(0)=5$ and $g(0)=0$, $C_2+2C_4=25$, $2C_2-C_4=0$

$\therefore C_2=5,\ C_4=10$

$\therefore$ $\boldsymbol{f(x)=e^x+4e^{-\frac{1}{2}x^2},\ g(x)=2e^x-2e^{-\frac{1}{2}x^2}}$

Differential Equations

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

➡ O165

1. Solve the following differential equation.

$$\frac{dy}{dx}=6x^2y \quad \cdots①$$

[Sol] (i) It is obvious that constant function $y=0$ is a solution.

(ii) When $y\neq0$, rearranging ①,

$$\frac{1}{y}\cdot\frac{dy}{dx}=6x^2$$

Integrating both sides with respect to x,

$$\int\frac{dy}{y}=\int 6x^2dx$$

$$\therefore\ \ln|y|=2x^3+C_1$$

$$\therefore\ y=\pm e^{2x^3+C_1}=\pm e^{C_1}\cdot e^{2x^3}$$

Let $\pm e^{C_1}=C$. $y=Ce^{2x^3}$ (C is an arbitrary constant, $C\neq0$)

When $C=0$ in (ii), $y=0$, which is the same as (i).

$\therefore\ y=Ce^{2x^3}$ **(C is an arbitrary constant)**

2. Find the particular solution to the following differential equation given the initial condition in the brackets [ ].

➡ O166

$$\frac{dy}{dx}=3y+1 \quad \cdots① \qquad [\text{When } x=1,\ y=1]$$

[Sol] Constant function $y=-\frac{1}{3}$ does not satisfy the given condition.

When $y\neq-\frac{1}{3}$, rearranging ①,

$$\frac{1}{3y+1}\cdot\frac{dy}{dx}=1$$

Integrating both sides with respect to x,

$$\int\frac{dy}{3y+1}=\int dx$$

$$\therefore\ \frac{1}{3}\ln|3y+1|=x+C_1$$

$$\therefore\ 3y+1=\pm e^{3x+3C_1}=\pm e^{3C_1}\cdot e^{3x}$$

Let $\pm e^{3C_1}=C$. $3y+1=Ce^{3x}$

When $x=1$, $y=1$; therefore, $C=4e^{-3}$

$\therefore\ 3y+1=4e^{3x-3}$ $\left[\ y=\frac{1}{3}(4e^{3x-3}-1)\right]$

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3. Solve the following differential equation using $x+y+1=u$. ➡ 0167

$$\frac{dy}{dx}=x+y+1 \quad \cdots①$$

[Sol] Differentiating both sides of $x+y+1=u$ with respect to x,

$$1+\frac{dy}{dx}=\frac{du}{dx} \quad \cdots②$$

From ① and ②, $\frac{du}{dx}=u+1 \quad \cdots③$

(i) When $u=-1$,

$y=-x-2$ and this satisfies ①; therefore this is a solution.

(ii) When $u\neq-1$, rearranging ③,

$$\frac{1}{u+1}\cdot\frac{du}{dx}=1$$

Integrating both sides with respect to x,

$$\int\frac{du}{u+1}=\int dx$$

$\therefore\ \ln|u+1|=x+C_1$

$\therefore\ u+1=\pm e^{x+C_1}=\pm e^{C_1}\cdot e^x$

Let $\pm e^{C_1}=C$. $u=Ce^x-1$ (C is an arbitrary constant, $C\neq0$)

When $C=0$ in (ii), $u=-1$, which is the same as (i).

$\therefore\ u=Ce^x-1$

$\therefore\ y=Ce^x-x-2$ **(C is an arbitrary constant)**

Natural/Social Science and Differential Equations

Name

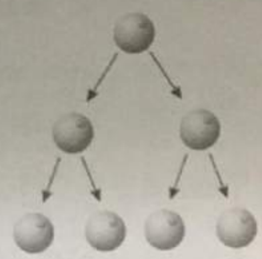
Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

Differential equations are widely applied to natural and social phenomena in daily life.

Ex. Certain bacteria divide and increase over time. Let the number of bacteria at time t be x. The rate at which the bacteria increase, i.e. $\frac{dx}{dt}$, is proportional to the number of the bacteria at any time. (Let the constant of proportionality be $k>0$.)



(1) Find the differential equation.

[Sol] $\frac{dx}{dt}=kx$ ← Since the bacteria divide and increase in number, the coefficient of x is positive.

(2) When the initial number of bacteria is 10, find the particular solution to the differential equation in question (1).

[Sol] From (1), $\frac{1}{x}\cdot\frac{dx}{dt}=k$ ← Considering $x\neq0$

Integrating both sides with respect to t,

$$\int\frac{dx}{x}=\int k\,dt$$

$$\therefore\ \ln|x|=kt+C_1$$

$$\therefore\ x=\pm e^{kt+C_1}=\pm e^{C_1}\cdot e^{kt}$$

Let $\pm e^{C_1}=C$. $x=Ce^{kt}$

When $t=0$, $x=10$; therefore, $C=\boxed{10}$

$$\therefore\ x=\boxed{10e^{kt}}$$

Answers: 10, $10e^{kt}$

0171b

Sugar begins to melt if it is put in hot water. Let the amount of unmelted sugar at time t be x. The rate at which sugar melts, i.e. $\frac{dx}{dt}$, is proportional to the amount of unmelted sugar at any time. (Let the constant of proportionality be $k > 0$.)

(1) Find the differential equation.

[Sol] $\frac{dx}{dt} = -kx$ ← Since the sugar melts and decreases in amount, the coefficient of x is negative.

(2) When the initial amount of sugar is 100, find the particular solution to the differential equation in question (1).

[Sol] From (1), $\frac{1}{x} \cdot \frac{dx}{dt} = -k$

Integrating both sides with respect to t,

$$\int \frac{dx}{x} = \int (-k)dt$$

$$\therefore \ln|x| = -kt + C_1$$

$$\therefore x = \pm e^{-kt+C_1} = \pm e^{C_1} \cdot e^{-kt}$$

Let $\pm e^{C_1} = C$. $x = Ce^{-kt}$

When $t=0$, $x=100$; therefore, $C=100$

$$\therefore x = 100e^{-kt}$$

Natural/Social Science and Differential Equations

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

Name

Date / /

Time : to :

Ex. When a cup of coffee heated to 90°C is left in a room with a temperature of 10°C, the temperature of the coffee begins to decrease. The rate at which the temperature of the coffee decreases is proportional to the difference in temperatures between the coffee and the room. Let the constant of proportionality be $k>0$ and assume that the room temperature stays the same.

(1) Let the temperature of the coffee after t minutes be x°C. Find the differential equation.

[Sol] $\dfrac{dx}{dt}=-k(x-10)$ ← The difference in temperatures between the coffee and the room after t minutes is $x-10$.

(2) Find the particular solution to the differential equation in question (1) given the initial condition.

[Sol] From (1), $\dfrac{1}{x-10}\cdot\dfrac{dx}{dt}=-k$

Integrating both sides with respect to t,

$$\int\frac{dx}{x-10}=\int(-k)dt$$

$\therefore \ln|x-10|=-kt+C_1 \quad \therefore x-10=\pm e^{-kt+C_1}=\pm e^{C_1}\cdot e^{-kt}$

Let $\pm e^{C_1}=C$. $x=Ce^{-kt}+10$

When $t=0$, $x=90$; therefore, $90=C+10$

$\therefore C=\boxed{80}$

$\therefore x=\boxed{80}\,e^{-kt}+10$

(3) After 3 minutes, the temperature of the coffee is 70°C. Find the temperature of the coffee if it is left in the room for another 3 minutes.

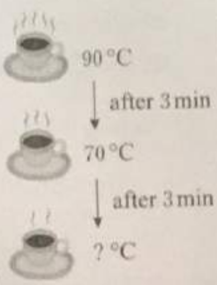
[Sol] When $t=3$, $x=70$; therefore, from (2),

$70=\boxed{80}\,e^{-3k}+10 \quad \therefore e^{-3k}=\boxed{\dfrac{3}{4}}$

Thus, since $t=6$ if it is left for another 3 minutes,

$x=\boxed{80}\,e^{-6k}+10$

$=\boxed{80}\,(e^{-3k})^2+10=\boxed{55}$ Ans. $\boxed{55}$ °C



Answers: 80, 80, 80, $\frac{3}{4}$, 80, 80, 55, 55

0172b

When a cup of tea heated to 80 °C is left in a room with a temperature of 20 °C, the temperature of the tea begins to decrease. The rate at which the temperature of the tea decreases is proportional to the difference in temperatures between the tea and the room. Let the constant of proportionality be $k > 0$ and assume that the room temperature stays the same.

(1) Let the temperature of the tea after t minutes be x°C. Find the differential equation.

[Sol] $\dfrac{dx}{dt} = -k(x-20)$

(2) Find the particular solution to the differential equation in question (1) given the initial condition.

[Sol] From (1), $\dfrac{1}{x-20} \cdot \dfrac{dx}{dt} = -k$

Integrating both sides with respect to t,

$$\int \frac{dx}{x-20} = \int (-k)\,dt$$

$\therefore \ \ln|x-20| = -kt + C_1$

$\therefore \ x-20 = \pm e^{-kt+C_1} = \pm e^{C_1} \cdot e^{-kt}$

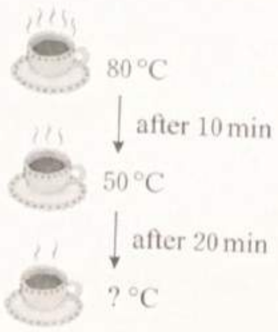
Let $\pm e^{C_1} = C$. $x = Ce^{-kt} + 20$

When $t=0$, $x=80$; therefore, $80 = C + 20$

$\therefore \ C = 60$

$\therefore \ \boldsymbol{x = 60e^{-kt} + 20}$

(3) After 10 minutes, the temperature of the tea is 50 °C. Find the temperature of the tea if it is left in the room for another 20 minutes. Write the answer in the decimal form.



[Sol] When $t=10$, $x=50$; therefore, from (2),

$50 = 60e^{-10k} + 20 \quad \therefore \ e^{-10k} = \dfrac{1}{2}$

Thus, since $t=30$ if it is left for another 20 minutes,

$x = 60e^{-30k} + 20$

$= 60(e^{-10k})^3 + 20$

$= 27.5$

Ans. 27.5 °C

Natural/Social Science and Differential Equations

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	1	—	2

Name

Date / /

Time : to :

1. When an object with a temperature of $100\,^\circ C$ is left in an area with a temperature of $20\,^\circ C$, the temperature of the object begins to decrease. The rate at which the temperature of the object decreases is proportional to the difference in temperatures between the object and the area. Let the constant of proportionality be $k>0$ and assume that the temperature of the area stays the same.

(1) Let the temperature of the object after t minutes be $x\,^\circ C$. Find the differential equation.

[Sol] $\dfrac{dx}{dt}=-k(x-20)$

(2) Find the particular solution to the differential equation in question (1) given the initial condition.

[Sol] From (1), $\dfrac{1}{x-20}\cdot\dfrac{dx}{dt}=-k$

Integrating both sides with respect to t,

$$\int\frac{dx}{x-20}=\int(-k)dt$$

$\therefore\ \ln|x-20|=-kt+C_1$

$\therefore\ x-20=\pm e^{-kt+C_1}=\pm e^{C_1}\cdot e^{-kt}$

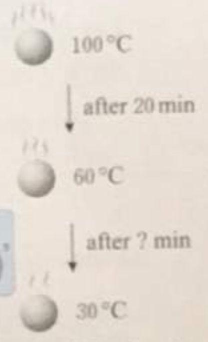
Let $\pm e^{C_1}=C$. $x=Ce^{-kt}+20$

When $t=0$, $x=100$; therefore, $100=C+20$

$\therefore\ C=80$

$\therefore\ x=80e^{-kt}+20$

(3) After 20 minutes, the temperature of the object is $60\,^\circ C$. Determine how many more minutes it takes for the object to become $30\,^\circ C$.



[Sol] When $t=20$, $x=60$; therefore, from (2),

$60=80e^{-20k}+20 \quad \therefore\ e^{-20k}=\dfrac{1}{2}$ …①

Let the time it takes for the object to become $30\,^\circ C$ be t_0 minutes.

$30=80e^{-kt_0}+20 \quad \therefore\ e^{-kt_0}=\dfrac{1}{8}=\left(\dfrac{1}{2}\right)^3$ …② From ①, $(e^{-20k})^3=\left(\dfrac{1}{2}\right)^3$

From ① and ②, $e^{-kt_0}=(e^{-20k})^3=e^{-60k}$

$\therefore\ t_0=60$ $\quad 60-20=40$ (min)

Therefore, it takes another 40 minutes.

Ans. **40 minutes**

O173b

2. The temperature of bathwater heated to 60 °C begins to decrease. The rate at which the temperature of the bathwater decreases is proportional to the difference in temperatures between the bathwater and the room. Let the constant of proportionality be $k>0$ and assume that the room temperature stays at 15 °C.

(1) Let the temperature of the bathwater after t minutes be x °C. Find the differential equation. Then, find the particular solution to it given the initial condition.

[Sol] $\dfrac{dx}{dt}=-k(x-15)$, i.e. $\dfrac{1}{x-15}\cdot\dfrac{dx}{dt}=-k$

Integrating both sides with respect to t,

$$\int\frac{dx}{x-15}=\int(-k)dt$$

$$\therefore\ \ln|x-15|=-kt+C_1$$

$$\therefore\ x-15=\pm e^{-kt+C_1}=\pm e^{C_1}\cdot e^{-kt}$$

Let $\pm e^{C_1}=C$. $x=Ce^{-kt}+15$

When $t=0$, $x=60$; therefore, $60=C+15$

$$\therefore\ C=45$$

$$\therefore\ \boldsymbol{x=45e^{-kt}+15}$$

(2) After 10 minutes, the temperature of the bathwater is 55 °C. Determine how many more minutes it takes for the bathwater to become 40 °C since it was heated. Use $\ln 2=0.6931$, $\ln 3=1.099$, $\ln 5=1.609$, and round off to the nearest integer.

[Sol] When $t=10$, $x=55$; therefore, from (1),

$$55=45e^{-10k}+15\quad\therefore\ e^{-10k}=\frac{8}{9}$$

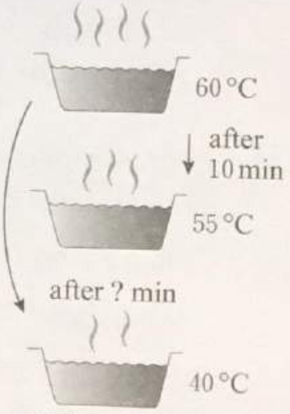
$$\therefore\ -10k=\ln\frac{8}{9}=3\ln 2-2\ln 3=-0.1187$$ ※

$$\therefore\ k=0.01187\ \cdots①$$

Let the time it takes for the bathwater to become 40 °C since it was heated be t_0 minutes.

$$40=45e^{-kt_0}+15\quad\therefore\ e^{-kt_0}=\frac{5}{9}$$

$$\therefore\ -kt_0=\ln\frac{5}{9}=\ln 5-2\ln 3=-0.589\ \cdots②$$ ※



From ① and ②, $t_0=\dfrac{0.589}{0.01187}=49.6\ldots\approx 50$

Ans. **50 minutes**

※ $\ln\dfrac{M}{N}=\ln M-\ln N$, $\ln M^n=n\ln M$

Natural/Social Science and Differential Equations

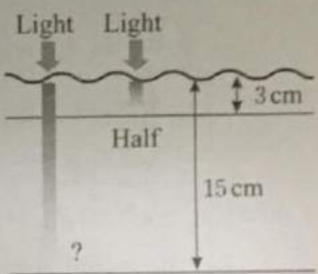
100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

Name

Date / /

Time : to :

1. Light does not reach the deep sea as it is absorbed when traveling through the liquid. The rate at which the amount of light changes is proportional to the amount of light that reaches a certain depth. (Let the constant of proportionality be $k>0$.) Given that half the amount of light is absorbed when traveling through the liquid with a depth of 3cm, find the remaining amount of light after it has traveled 15cm. Let the amount of light x cm below the liquid surface be y, and the amount of light at the liquid surface be y_0.



[Sol] $\dfrac{dy}{dx}=-ky$, i.e. $\dfrac{1}{y}\cdot\dfrac{dy}{dx}=-k$

Integrating both sides with respect to x,

$$\int\frac{dy}{y}=\int(-k)dx$$

$$\therefore \ \ln|y|=-kx+C_1$$

$$\therefore \ y=\pm e^{-kx+C_1}=\pm e^{C_1}\cdot e^{-kx}$$

Let $\pm e^{C_1}=C$. $y=Ce^{-kx}$

When $x=0$, $y=y_0$; therefore, $C=y_0$

The amount of light at the liquid surface ($x=0$) is y_0.

$$\therefore \ y=y_0e^{-kx}$$

Also, when $x=3$, $y=\dfrac{1}{2}y_0$; therefore, $\dfrac{1}{2}y_0=y_0e^{-3k}$

$$\therefore \ e^{-3k}=\frac{1}{2}$$

Half the amount of light is absorbed when traveling through the liquid with a depth of 3cm.

Thus, the remaining amount of light traveling 15cm is

Substituting $x=15$ into $y=y_0e^{-kx}$

$$y=y_0e^{-15k}$$
$$=y_0(e^{-3k})^5$$
$$=\frac{1}{32}y_0$$

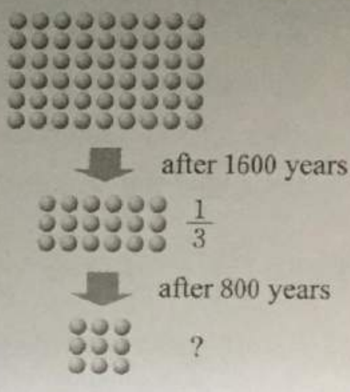
Ans. $\dfrac{1}{32}y_0$

0174b

2. A certain substance Z decays over time while emitting radiation. The rate at which Z decays is proportional to the remaining amount of Z. (Let the constant of proportionality be $k > 0$.) Given that $\frac{1}{3}$ of the initial amount of Z remains after 1600 years, find how much of the initial amount will remain after another 800 years.

[Sol] Let the remaining amount of Z after t years be x.

$$\frac{dx}{dt} = -kx, \text{ i.e. } \frac{1}{x} \cdot \frac{dx}{dt} = -k$$



Integrating both sides with respect to t,

$$\int \frac{dx}{x} = \int (-k)\,dt$$

$$\therefore \ \ln|x| = -kt + C_1$$

$$\therefore \ x = \pm e^{-kt+C_1} = \pm e^{C_1} \cdot e^{-kt}$$

Let $\pm e^{C_1} = C$. $x = Ce^{-kt}$

Let the initial amount of Z be x_0. $C = x_0$ ⟵ When $t=0$, $x=x_0$

$$\therefore \ x = x_0 e^{-kt}$$

Also, when $t=1600$, $x=\frac{1}{3}x_0$; therefore, $\frac{1}{3}x_0 = x_0 e^{-1600k}$

$$\therefore \ e^{-1600k} = \frac{1}{3}$$

Thus, since $t=2400$ after another 800 years,

$$x = x_0 e^{-2400k}$$

$$= x_0 \left(e^{-1600k}\right)^{\frac{3}{2}}$$

$$= \frac{\sqrt{3}}{9} x_0$$

$\left(\frac{1}{3}\right)^{\frac{3}{2}} = \frac{1}{3\sqrt{3}} = \frac{\sqrt{3}}{9}$

Ans. $\frac{\sqrt{3}}{9}$ **of the initial amount**

O175a

KUMON

Natural/Social Science and Differential Equations

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

1. Air has weight. The pressure to the ground exerted by the weight of air is called atmospheric pressure, and it decreases as the altitude increases. The rate at which atmospheric pressure changes is proportional to the pressure at a given altitude. (Let the constant of proportionality be $k > 0$.) Given that atmospheric pressure at an altitude of 5500 m is 50% of the pressure at sea level (an altitude of 0 m), find the pressure at an altitude of 8800 m as a percentage of the pressure at sea level. Use $2^{\frac{3}{5}} = 1.515$, and round off to the nearest integer.

[Sol] Let the pressure at an altitude of h m be p.

$$\frac{dp}{dh} = -kp, \text{ i.e. } \frac{1}{p}\cdot\frac{dp}{dh} = -k$$

Integrating both sides with respect to h,

$$\int \frac{dp}{p} = \int (-k)\,dh$$

$$\therefore\ \ln|p| = -kh + C_1$$

$$\therefore\ p = \pm e^{-kh+C_1} = \pm e^{C_1}\cdot e^{-kh}$$

Let $\pm e^{C_1} = C$. $p = Ce^{-kh}$

Let the pressure at sea level be p_0. $C = p_0$

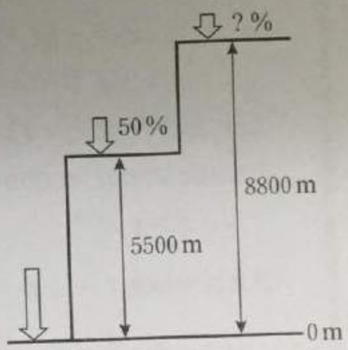
$$\therefore\ p = p_0 e^{-kh}$$

Also, when $h = 5500$, $p = \frac{1}{2}p_0$; therefore, $\frac{1}{2}p_0 = p_0 e^{-5500k}$

$$\therefore\ e^{-5500k} = \frac{1}{2}$$

Thus, atmospheric pressure at an altitude of 8800 m is

$$p = p_0 e^{-8800k}$$
$$= p_0 (e^{-5500k})^{\frac{8}{5}}$$
$$= \frac{1}{2\times 1.515}p_0 \qquad \boxed{\left(\frac{1}{2}\right)^{\frac{8}{5}} = \frac{1}{2\cdot 2^{\frac{3}{5}}}}$$
$$= \frac{1}{3.03}p_0$$
$$\approx 0.33 p_0$$

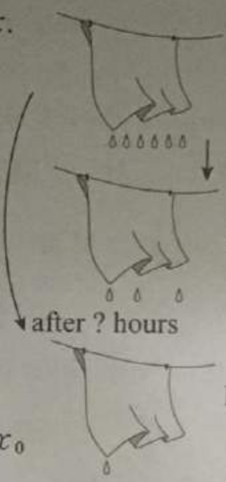


Ans. 33 %

O175b

2. In certain weather conditions, a wet object loses its moisture. The rate of moisture loss is proportional to the amount of moisture at any time. (Let the constant of proportionality be $k > 0$.) Given that a wet bed sheet loses half of its moisture in 1 hour, determine how many hours it takes for the bed sheet to lose 99% of its moisture. Assume that weather conditions stay the same. Use $\log_2 5 = 2.32$, and round off to one decimal place.

[Sol] Let the amount of moisture after t hours be x.



$$\frac{dx}{dt} = -kx, \text{ i.e. } \frac{1}{x} \cdot \frac{dx}{dt} = -k$$

Integrating both sides with respect to t,

$$\int \frac{dx}{x} = \int (-k)\,dt$$

$$\therefore \ \ln|x| = -kt + C_1$$

$$\therefore \ x = \pm e^{-kt+C_1} = \pm e^{C_1} \cdot e^{-kt}$$

Let $\pm e^{C_1} = C$. $x = Ce^{-kt}$

Let the initial amount of moisture be x_0. $C = x_0$

$$\therefore \ x = x_0 e^{-kt}$$

Also, when $t=1$, $x = \frac{1}{2}x_0$; therefore, $\frac{1}{2}x_0 = x_0 e^{-k}$

$$\therefore \ e^{-k} = \frac{1}{2}$$

Thus, let the time it takes for the bed sheet to lose 99% of its moisture be t_0 hours.

$$\frac{1}{100}x_0 = x_0 e^{-kt_0}$$

Losing 99% of its moisture, the amount of moisture remained is $\frac{1}{100}x_0$.

$$= x_0 (e^{-k})^{t_0}$$

$$= x_0 \left(\frac{1}{2}\right)^{t_0}$$

Since $\frac{1}{100} = \frac{1}{2^{t_0}}$, $2^{t_0} = 100$

$\therefore \ t_0 = \log_2 100$

Therefore,

$$t_0 = \log_2 100$$

$$= 2(1 + \log_2 5)$$

$\log_a MN = \log_a M + \log_a N$

$\log_a M^n = n \log_a M$

$$= 6.64$$

$$\approx 6.6$$

Ans. **6.6 hours**

O176a

KUMON

Natural/Social Science and Differential Equations

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
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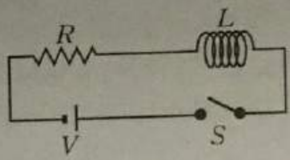
Ex. The strength of an electrical current x flowing through the electric circuit shown on the right is expressed as a function of time t and satisfies the following differential equation.



$$L\frac{dx}{dt}+Rx=V$$

Find the strength of the electrical current x after switch S is closed at time $t=0$. Assume that L, R and V are positive constants, and also that $x=0$ when $t=0$. Then, find $\lim_{t\to\infty} x$.

[Sol] Since $L\frac{dx}{dt}+Rx=V$, $\frac{1}{V-Rx}\cdot\frac{dx}{dt}=\frac{1}{L}$

Integrating both sides with respect to t,

$$\int\frac{dx}{V-Rx}=\int\frac{dt}{L}$$

$$\therefore\ -\frac{1}{R}\ln|V-Rx|=\frac{1}{L}t+C_1$$

$$\therefore\ V-Rx=\pm e^{-R\left(\frac{1}{L}t+C_1\right)}=\pm e^{-RC_1}\cdot e^{-\frac{R}{L}t}$$

Let $\pm e^{-RC_1}=C$. $V-Rx=Ce^{-\frac{R}{L}t}$

When $t=0$, $x=0$; therefore, $C=\boxed{V}$

$$\therefore\ V-Rx=\boxed{V}\,e^{-\frac{R}{L}t}$$

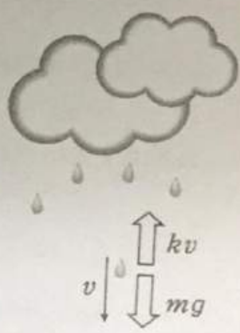
$$\therefore\ x=\boxed{\frac{V}{R}}\left(1-e^{-\frac{R}{L}t}\right)$$

Also, $\lim_{t\to\infty} x=\lim_{t\to\infty}\boxed{\frac{V}{R}}\left(1-e^{-\frac{R}{L}t}\right)=\boxed{\frac{V}{R}}$ ⟵ $\lim_{t\to\infty} e^{-\frac{R}{L}t}=0$

Answers: V, V, $\frac{V}{R}$, $\frac{V}{R}$, $\frac{V}{R}$

0176b

1. Raindrops do not exceed a certain speed because of air resistance. When an object falls in the air, it is subject to air resistance which is proportional to its speed. If v is the speed of the object with mass m at time t, $m\dfrac{dv}{dt}=mg-kv$ is true. (g represents the acceleration due to gravity and is a positive constant, and k is also a positive constant.) Given that the initial speed is 0, solve this differential equation. Then, find $\lim_{t\to\infty} v$ representing the speed of the raindrop with mass m falling to the ground.



[Sol] Since $m\dfrac{dv}{dt}=mg-kv$, $\dfrac{1}{mg-kv}\cdot\dfrac{dv}{dt}=\dfrac{1}{m}$

Integrating both sides with respect to t,

$$\int\frac{dv}{mg-kv}=\int\frac{dt}{m}$$

$$\therefore\ -\frac{1}{k}\ln|mg-kv|=\frac{1}{m}t+C_1$$

$$\therefore\ mg-kv=\pm e^{-k\left(\frac{1}{m}t+C_1\right)}=\pm e^{-kC_1}\cdot e^{-\frac{k}{m}t}$$

Let $\pm e^{-kC_1}=C$. $mg-kv=Ce^{-\frac{k}{m}t}$

When $t=0$, $v=0$; therefore, $C=mg$

$$\therefore\ mg-kv=mge^{-\frac{k}{m}t}$$

$$\therefore\ v=\frac{mg}{k}\left(1-e^{-\frac{k}{m}t}\right)$$

Also, $\lim_{t\to\infty} v=\lim_{t\to\infty}\dfrac{mg}{k}\left(1-e^{-\frac{k}{m}t}\right)=\dfrac{mg}{k}$

Natural/Social Science and Differential Equations

Name

Date / /

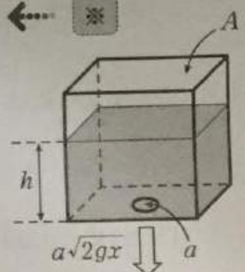
Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

Ex. There is a small hole with area a at the bottom of a container filled with water. The amount of water drained from the hole per second is $a\sqrt{2gx}$. (x represents the height of water.) The cross-section area A of the container is always constant. Let the initial height of water be h. Determine how many seconds it takes for the height of water to become $\frac{h}{2}$. Assume that g represents the acceleration due to gravity and is a positive constant.

[Sol] Let the height of water after t seconds be x.

$$A\frac{dx}{dt}=-a\sqrt{2gx}, \text{ i.e. } \frac{1}{\sqrt{x}}\cdot\frac{dx}{dt}=-\frac{a\sqrt{2g}}{A} \quad \leftarrow ※$$



Integrating both sides with respect to t,

$$\int\frac{dx}{\sqrt{x}}=\int\left(-\frac{a\sqrt{2g}}{A}\right)dt$$

$$\therefore\ 2\sqrt{x}=-\frac{a\sqrt{2g}}{A}t+C$$

When $t=0$, $x=h$; therefore, $C=2\boxed{\sqrt{h}}$

$$\therefore\ 2\sqrt{x}=-\frac{a\sqrt{2g}}{A}t+2\boxed{\sqrt{h}}$$

Substituting $x=\frac{h}{2}$ into the equation above

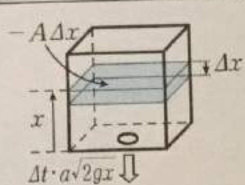
Thus, the time it takes for the height of water to become $\frac{h}{2}$ is, since $2\sqrt{\frac{h}{2}}=-\frac{a\sqrt{2g}}{A}t+2\boxed{\sqrt{h}}$,

$$t=\frac{A}{a}\sqrt{\frac{h}{g}}\left(\boxed{\sqrt{2}-1}\right)$$

Ans. $\frac{A}{a}\sqrt{\frac{h}{g}}\left(\boxed{\sqrt{2}-1}\right)$ seconds

Answers: $\sqrt{h}$, $\sqrt{h}$, $\sqrt{h}$, $\sqrt{h}$, $\sqrt{2}-1$, $\sqrt{2}-1$, $\sqrt{2}-1$

※ It is possible to consider the following:
If the height of water changes for Δx from t to $t+\Delta t$, the amount of water decreased in the container is $-A\Delta x$, and the amount of water drained from the hole is $\Delta t\cdot a\sqrt{2gx}$



$$\therefore\ -A\Delta x=\Delta t\cdot a\sqrt{2gx} \quad \therefore\ A\frac{\Delta x}{\Delta t}=-a\sqrt{2gx}$$

Then, let $\Delta t\to 0$. Since $\frac{\Delta x}{\Delta t}\to\frac{dx}{dt}$, $A\frac{dx}{dt}=-a\sqrt{2gx}$

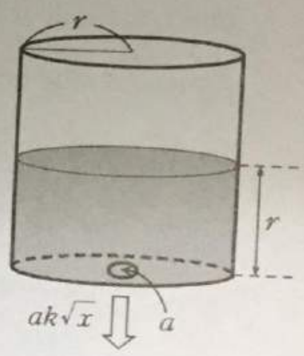
0177b

1. There is a small hole with area a at the bottom of a container filled with water. The amount of water drained from the hole per second is $ak\sqrt{x}$. (k is a constant of proportionality, and $k > 0$ and x represents the height of water.) The container is a cylinder with base radius r, and the initial height of water is r. Find the time necessary for all the water to be drained from this container.

[Sol] Let the height of water after t seconds be x.

$$\pi r^2 \frac{dx}{dt} = -ak\sqrt{x}, \text{ i.e. } \frac{1}{\sqrt{x}} \cdot \frac{dx}{dt} = -\frac{ak}{\pi r^2}$$

Integrating both sides with respect to t,



$$\int \frac{dx}{\sqrt{x}} = \int \left(-\frac{ak}{\pi r^2}\right) dt$$

$$\therefore\ 2\sqrt{x} = -\frac{ak}{\pi r^2} t + C$$

When $t=0$, $x=r$; therefore, $C=2\sqrt{r}$

$$\therefore\ 2\sqrt{x} = -\frac{ak}{\pi r^2} t + 2\sqrt{r}$$

Thus, the time necessary for all the water to be drained from this container is, since $0 = -\frac{ak}{\pi r^2} t + 2\sqrt{r}$, ← Substituting $x=0$ into the equation above

$$t = \frac{2\pi r^2 \sqrt{r}}{ak}$$

Ans. $\frac{2\pi r^2\sqrt{r}}{ak}$ seconds

Natural/Social Science and Differential Equations

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	1	2~

1. The air in a room with volume 900m^3 contains 0.12% of Carbon Dioxide (CO_2). Fresh air containing 0.04% of CO_2 flows into the room. Find the amount of inflowing fresh air per minute required to decrease the percentage of CO_2 to 0.06% in 10 minutes. Assume that the existing air is immediately mixed with fresh air evenly, and also that the amount of inflowing air is equal to the amount of outflowing air. Use $\ln 2=0.693$, and round off to the nearest integer.

[Sol] Let the amount of CO_2 after t minutes be $x\text{m}^3$, and the amount of inflowing air per minute be $a\text{m}^3$.

$$\frac{dx}{dt}=\frac{0.04}{100}a-\frac{x}{900}a, \text{ i.e. } \frac{1}{25x-9}\cdot\frac{dx}{dt}=-\frac{a}{22500}$$ ※

Integrating both sides with respect to t,

$$\int\frac{dx}{25x-9}=\int\left(-\frac{a}{22500}\right)dt$$

$$\therefore\ \frac{1}{25}\ln|25x-9|=-\frac{a}{22500}t+C_1$$

$$\therefore\ 25x-9=\pm e^{25\left(-\frac{a}{22500}t+C_1\right)}=\pm e^{25C_1}\cdot e^{-\frac{a}{900}t}$$

Let $\pm e^{25C_1}=C$. $25x-9=Ce^{-\frac{a}{900}t}$

When $t=0$, $x=900\cdot\frac{0.12}{100}=\frac{27}{25}$; therefore, ← 0.12% of 900m^3

$$C=25\cdot\frac{27}{25}-9=18$$

$$\therefore\ 25x-9=18e^{-\frac{a}{900}t}$$

Also, when $t=10$, $x=900\cdot\frac{0.06}{100}=\frac{27}{50}$; therefore, ← 0.06% of 900m^3

$$25\cdot\frac{27}{50}-9=18e^{-\frac{a}{900}\cdot 10}$$

$\ln\frac{1}{N}=-\ln N,\ \ \ln M^n=n\ln M$

$$\therefore\ e^{-\frac{a}{90}}=\frac{1}{4}\quad\therefore\ -\frac{a}{90}=\ln\frac{1}{4}=-2\ln 2$$

$$\therefore\ a=180\ln 2=124.74\approx 125$$

Ans. 125 m^3

※ From t to $t+\Delta t$, the amount of CO_2 contained is $\Delta t\cdot\frac{0.04}{100}a$ for inflow and $\Delta t\cdot\frac{x}{900}a$ for outflow.

O178b

2. A certain piece of information is spread in a town with a population of N people. (N is a constant.) Information is spread when one person who has the information shares it with another person who does not. Therefore, let the number of people who have the information be a function of the time $n=n(t)$, and let the rate of information spreading, i.e. the rate of increase in the number of people who have the information $\frac{dn}{dt}$, be expressed as $\frac{dn}{dt}=kn(N-n)$. (k is a positive constant.) Solve the following questions.

(1) Let $y=\frac{n}{N-n}$. Express $\frac{dy}{dt}$ without using n.

[Sol] $\frac{dy}{dt}=\frac{n'(N-n)-n\cdot(-n')}{(N-n)^2}=\frac{n'N}{(N-n)^2}$

$\therefore\ \frac{dy}{dt}=\frac{N}{(N-n)^2}\cdot kn(N-n)=\frac{knN}{N-n}=kNy$ $n'=\frac{dn}{dt}$

$\therefore\ \frac{dy}{dt}=kNy$ $\frac{n}{N-n}=y$

(2) If half of the population in the town has the information at $t=0$, find y.

[Sol] From (1), $\frac{1}{y}\cdot\frac{dy}{dt}=kN$

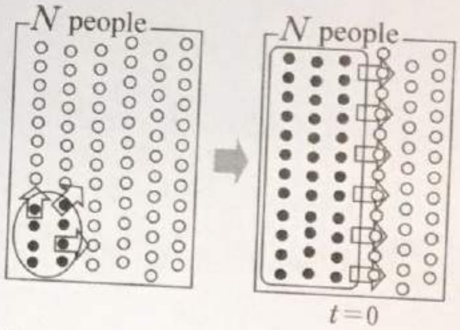
Integrating both sides with respect to t,

$\int\frac{dy}{y}=\int kNdt$

$\therefore\ \ln|y|=kNt+C_1$

$\therefore\ y=\pm e^{kNt+C_1}=\pm e^{C_1}\cdot e^{kNt}$

Let $\pm e^{C_1}=C$. $y=Ce^{kNt}$



When $t=0$, $n=\frac{1}{2}N$; therefore, $y=\frac{\frac{1}{2}N}{N-\frac{1}{2}N}=1$

Substituting $n=\frac{1}{2}N$ into $y=\frac{n}{N-n}$

$\therefore\ C=1$ Substituting $t=0$ and $y=1$ into $y=Ce^{kNt}$

$\therefore\ y=e^{kNt}$

(3) Using y found in question (2), find $\lim_{t\to\infty} n$.

$y=\frac{n}{N-n}$

[Sol] From (2), $\frac{n}{N-n}=e^{kNt}$; therefore, $n=\frac{Ne^{kNt}}{e^{kNt}+1}$

$\therefore\ \lim_{t\to\infty} n=\lim_{t\to\infty}\frac{Ne^{kNt}}{e^{kNt}+1}=\lim_{t\to\infty}\frac{N}{1+e^{-kNt}}=N$

It can be said that all the people in this town would eventually have this information after an extremely long time.

Natural/Social Science and Differential Equations

Name

Date / /

Time : to :

100%	~90%	~80%	~70%	69%~
(mistakes) 0	—	—	—	1~

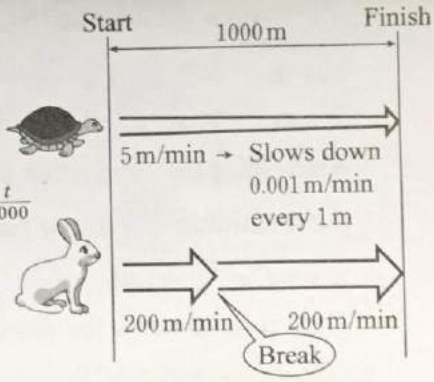
1. A rabbit and a turtle had a 1000m race. The turtle started walking at a speed of 5m/min. He kept walking without a break, but his speed continuously decreased at a rate of 0.001m/min every 1m. On the other hand, the rabbit kept running at a speed of 200m/min but took a break in the middle. As a result, the turtle finished the race 1 minute earlier than the rabbit. Find how many minutes the rabbit took for a break. Use $\ln 2=0.693$ and $\ln 5=1.609$.

[Sol] Let the distance that the turtle walked in t minutes be s m.

$$\frac{ds}{dt}=5-\frac{1}{1000}s\text{, i.e. }\frac{1}{s-5000}\cdot\frac{ds}{dt}=-\frac{1}{1000}$$

$\frac{ds}{dt}$ represents the speed.

Integrating both sides with respect to t,



$$\int\frac{ds}{s-5000}=\int\left(-\frac{1}{1000}\right)dt$$

$$\therefore\ \ln|s-5000|=-\frac{t}{1000}+C_1$$

$$\therefore\ s-5000=\pm e^{-\frac{t}{1000}+C_1}=\pm e^{C_1}\cdot e^{-\frac{t}{1000}}$$

Let $\pm e^{C_1}=C$. $s=Ce^{-\frac{t}{1000}}+5000$

When $t=0$, $s=0$; therefore, $C=-5000$

$$\therefore\ s=-5000\left(e^{-\frac{t}{1000}}-1\right)$$

Let the time that the turtle took to finish the race be t_0 minutes.

$$1000=-5000\left(e^{-\frac{t_0}{1000}}-1\right)$$

When $t=t_0$, $s=1000$

$$\therefore\ e^{-\frac{t_0}{1000}}=\frac{4}{5}\qquad\therefore\ -\frac{t_0}{1000}=\ln\frac{4}{5}=2\ln 2-\ln 5$$

The time while the rabbit was running: $t_0+1-(\text{break})$

$$\therefore\ t_0=-1000(2\ln 2-\ln 5)=223$$

Let the time of the break that the rabbit took be x minutes.

$$200(223+1-x)=1000\qquad\therefore\ x=219$$

Ans. **219 minutes**

0179b

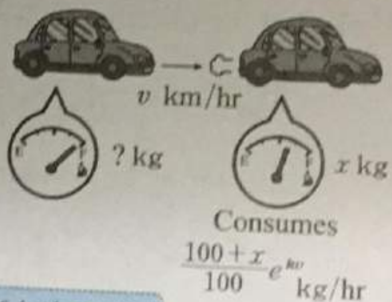
2. A car filled with $x$kg of gasoline consumes $\dfrac{100+x}{100}e^{kv}$kg/hr of gasoline when driving at $v$km/hr. (k is a positive constant.) Given that this car drives to a destination 100km away at a constant speed, find the initial amount of gasoline and the driving speed such that gasoline consumption is minimized. Assume that the car stops immediately after running out of gasoline.

[Sol] Let the remaining amount of gasoline after t hours while driving at $v$km/hr be $x$kg.

$$\frac{dx}{dt}=-\frac{100+x}{100}e^{kv}, \text{ i.e. } \frac{1}{100+x}\cdot\frac{dx}{dt}=-\frac{e^{kv}}{100}$$

Integrating both sides with respect to t,

$$\int\frac{dx}{100+x}=\int\left(-\frac{e^{kv}}{100}\right)dt$$



$$\therefore\ \ln|100+x|=-\frac{e^{kv}}{100}t+C_1$$

$$\therefore\ 100+x=\pm e^{-\frac{e^{kv}}{100}t+C_1}=\pm e^{C_1}\cdot e^{-\frac{e^{kv}}{100}t}$$

Let $\pm e^{C_1}=C$. $x=Ce^{-\frac{e^{kv}}{100}t}-100$

When $t=0$, $x=C-100$ $\cdots$①

$x=C-100$ is the initial amount of gasoline.

Also, since the time taken to drive 100km is $t=\dfrac{100}{v}$, the remaining amount of gasoline when the car arrives at the destination is $x=Ce^{-\frac{e^{kv}}{v}}-100$.

Therefore, to reach the destination 100km away,

$Ce^{-\frac{e^{kv}}{v}}-100\geq 0$, i.e. $C\geq 100e^{\frac{e^{kv}}{v}}$ is true. ※

Let $f(v)=\dfrac{e^{kv}}{v}$. $f'(v)=\dfrac{ke^{kv}\cdot v-e^{kv}\cdot 1}{v^2}=\dfrac{e^{kv}(kv-1)}{v^2}$

When $f'(v)=0$ in $v>0$, $v=\dfrac{1}{k}$

v	0	$\cdots$	$\frac{1}{k}$	$\cdots$
$f'(v)$		$-$	0	$+$
$f(v)$		$\searrow$	ke	$\nearrow$

From the variation table, $f(v)$ is a minimum at $v=\dfrac{1}{k}$.

Then, the minimum value of C is $100e^{ke}$.

Thus, from ①, the initial amount of gasoline should be,

$x=100e^{ke}-100=100(e^{ke}-1)$

Substituting $\frac{e^{kv}}{v}=ke$ into the inequality denoted by ※

Ans. **The initial amount of gasoline: $100(e^{ke}-1)$ kg**

The driving speed: $\dfrac{1}{k}$ km/hr

Natural/Social Science and Differential Equations

Name ______ Date / / Time : to :

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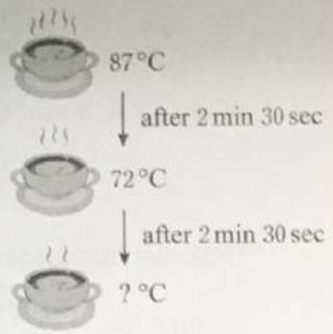
1. When a bowl of soup heated to 87°C is left in a room with a temperature of 12°C for 2 minutes and 30 seconds, the temperature of the soup becomes 72°C. Find the temperature of the soup if it is left in the room for another 2 minutes and 30 seconds given that the rate at which the temperature of the soup decreases is proportional to the difference in temperatures between the soup and the room. Let the constant of proportionality be $k > 0$ and assume that the room temperature stays the same.

➡ O172

[Sol] Let the temperature of the soup after t minutes be x°C.

$$\frac{dx}{dt} = -k(x-12), \text{ i.e. } \frac{1}{x-12}\cdot\frac{dx}{dt} = -k$$

Integrating both sides with respect to t,



$$\int \frac{dx}{x-12} = \int (-k)\,dt$$

$$\therefore\ \ln|x-12| = -kt + C_1$$

$$\therefore\ x-12 = \pm e^{-kt+C_1} = \pm e^{C_1}\cdot e^{-kt}$$

Let $\pm e^{C_1} = C$. $x = Ce^{-kt} + 12$

When $t=0$, $x=87$; therefore, $87 = C + 12$

$$\therefore\ C = 75$$

$$\therefore\ x = 75e^{-kt} + 12$$

Also, when $t = \frac{5}{2}$, $x = 72$; therefore, $72 = 75e^{-\frac{5}{2}k} + 12$

$$\therefore\ e^{-\frac{5}{2}k} = \frac{4}{5}$$

Thus, since $t=5$ when the soup is left for another 2 minutes and 30 seconds,

$$\begin{aligned} x &= 75e^{-5k} + 12 \\ &= 75\left(e^{-\frac{5}{2}k}\right)^2 + 12 \\ &= 60 \end{aligned}$$

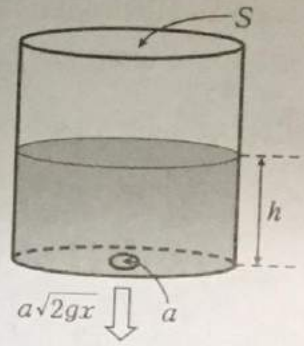
Ans. **60 °C**

O180b

2. There is a small hole with area a at the bottom of a container filled with water. The amount of water drained from the hole per second is $a\sqrt{2gx}$. (x represents the height of water.) The cross-section area S of the container is always constant. Let the initial height of water be h. Find the time necessary for all the water to be drained from this container. Assume that g represents the acceleration due to gravity and is a positive constant. ➡ O177

[Sol] Let the height of water after t seconds be x.

$$S\frac{dx}{dt}=-a\sqrt{2gx}, \text{ i.e. } \frac{1}{\sqrt{x}}\cdot\frac{dx}{dt}=-\frac{a\sqrt{2g}}{S}$$



Integrating both sides with respect to t,

$$\int\frac{dx}{\sqrt{x}}=\int\left(-\frac{a\sqrt{2g}}{S}\right)dt$$

$$\therefore\ 2\sqrt{x}=-\frac{a\sqrt{2g}}{S}t+C$$

When $t=0$, $x=h$; therefore, $C=2\sqrt{h}$

$$\therefore\ 2\sqrt{x}=-\frac{a\sqrt{2g}}{S}t+2\sqrt{h}$$

Thus, the time necessary for all the water to be drained from this container is, since $0=-\frac{a\sqrt{2g}}{S}t+2\sqrt{h}$,

$$t=\frac{S}{a}\sqrt{\frac{2h}{g}}$$

Ans. $\frac{S}{a}\sqrt{\frac{2h}{g}}$ **seconds**

Applications of Calculus 1

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1. Given that function $f(x)=ae^{ax}+be^{-bx}$ satisfies $f(0)=1$ and $f'(0)+f''(0)=0$, solve the following questions. (a and b are constants and $a\neq0$.)

(1) Find the values of a and b.

[Sol] $f'(x)=a^2e^{ax}-b^2e^{-bx}$, $f''(x)=a^3e^{ax}+b^3e^{-bx}$

Since $f(0)=1$, $a+b=1$ …①

Since $f'(0)+f''(0)=0$, $a^2-b^2+a^3+b^3=0$

$\therefore\ (a+b)(a-b+a^2-ab+b^2)=0$ …②

$a^2-b^2=(a+b)(a-b)$
$a^3+b^3=(a+b)(a^2-ab+b^2)$

From ① and ②, $a-b+a^2-ab+b^2=0$

Also, substituting $b=1-a$,

$a-(1-a)+a^2-a(1-a)+(1-a)^2=0$

$\therefore\ a(3a-1)=0$

Since $a\neq0$, $\boldsymbol{a=\frac{1}{3}}$

Since $b=1-a$, $\boldsymbol{b=\frac{2}{3}}$

(2) Find the relative extreme values of the function $f(x)$.

➡ O13

[Sol] From (1),

$f(x)=\frac{1}{3}e^{\frac{1}{3}x}+\frac{2}{3}e^{-\frac{2}{3}x}$, $f'(x)=\frac{1}{9}e^{\frac{1}{3}x}-\frac{4}{9}e^{-\frac{2}{3}x}=\frac{1}{9}e^{-\frac{2}{3}x}(e^x-4)$

When $f'(x)=0$, $e^x=4$, i.e. $x=\ln4$

Creating the variation table,

x	…	$\ln4$	…
$f'(x)$	−	0	+
$f(x)$	↘	$\frac{\sqrt[3]{4}}{2}$	↗

Since $e^{\frac{1}{3}\ln4}=(e^{\ln4})^{\frac{1}{3}}=4^{\frac{1}{3}}=\sqrt[3]{4}$ and

$e^{-\frac{2}{3}\ln4}=(e^{\ln4})^{-\frac{2}{3}}=4^{-\frac{2}{3}}=\frac{1}{(\sqrt[3]{4})^2}=\frac{\sqrt[3]{4}}{4}$,

$f(\ln4)=\frac{1}{3}e^{\frac{1}{3}\ln4}+\frac{2}{3}e^{-\frac{2}{3}\ln4}=\frac{\sqrt[3]{4}}{3}+\frac{2}{3}\cdot\frac{\sqrt[3]{4}}{4}$

$=\frac{\sqrt[3]{4}}{3}+\frac{\sqrt[3]{4}}{6}=\frac{\sqrt[3]{4}}{2}$

Therefore, **the relative minimum value is $\frac{\sqrt[3]{4}}{2}$, at $x=\ln4$ and there are no relative maximum values.** $\boxed{\ln4=2\ln2}$

O181b

2. Let Q be the point of intersection of line $x=\pi a$ and the normal at point P on curve $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ $(0\leq\theta\leq 2\pi)$. (a is a positive constant and point P is not point $(\pi a, 2a)$.)

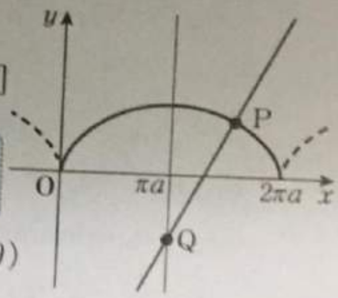
(1) Express the y-coordinate of Q in terms of θ. ➡ O7

[Sol] $\dfrac{dx}{d\theta}=a(1-\cos\theta)$, $\dfrac{dy}{d\theta}=a\sin\theta$

$$\therefore \frac{dy}{dx}=\frac{a\sin\theta}{a(1-\cos\theta)}=\frac{\sin\theta}{1-\cos\theta}$$ ⟵ $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Therefore, the equation of the normal at point P is

$$y-a(1-\cos\theta)=-\frac{1-\cos\theta}{\sin\theta}[x-a(\theta-\sin\theta)]$$



Thus, the y-coordinate of point Q is

Substituting $x=\pi a$

$$y=-\frac{1-\cos\theta}{\sin\theta}[\pi a-a(\theta-\sin\theta)]+a(1-\cos\theta)$$
$$=-\frac{a(\pi-\theta)(1-\cos\theta)}{\sin\theta}$$

(2) When θ approaches π, find the point which Q approaches. ➡ N126

[Sol] From (1), $y=-a[1+\cos(\pi-\theta)]\cdot\dfrac{\pi-\theta}{\sin(\pi-\theta)}$ ⟵ $\cos\theta=-\cos(\pi-\theta)$, $\sin\theta=\sin(\pi-\theta)$

Let $\pi-\theta=t$. As $\theta\to\pi$, $t\to 0$

$$\therefore \lim_{\theta\to\pi} y=\lim_{t\to 0}\left[-a(1+\cos t)\cdot\frac{t}{\sin t}\right]$$
$$=\lim_{t\to 0}\left[-a(1+\cos t)\cdot\frac{1}{\frac{\sin t}{t}}\right]$$
$$=-2a$$

$\displaystyle\lim_{x\to 0}\frac{\sin x}{x}=1$

Therefore, point Q approaches $(\pi a, -2a)$. ⟵ Since point Q is the point of intersection of the normal at point P and $x=\pi a$, the x-coordinate is πa.

Applications of Calculus 1

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Given function $f(x)=(x+2)e^{\frac{1}{x}}\ (x\neq 0)$, solve the following questions.

(1) Determine where the function increases/decreases, its relative extreme values, concavity and inflection point(s) of $y=f(x)\ (x\neq 0)$. Then, sketch its graph.

➡ O23

[Sol] $f'(x)=1\cdot e^{\frac{1}{x}}+(x+2)e^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)=\frac{(x+1)(x-2)e^{\frac{1}{x}}}{x^2}$

$$f''(x)=\frac{\left[(2x-1)e^{\frac{1}{x}}+(x+1)(x-2)e^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)\right]x^2-(x+1)(x-2)e^{\frac{1}{x}}\cdot 2x}{x^4}$$

$$=\frac{(5x+2)e^{\frac{1}{x}}}{x^4}$$

When $f'(x)=0$, $x=-1,\ 2$. When $f''(x)=0$, $x=-\frac{2}{5}$.

Creating the variation table,

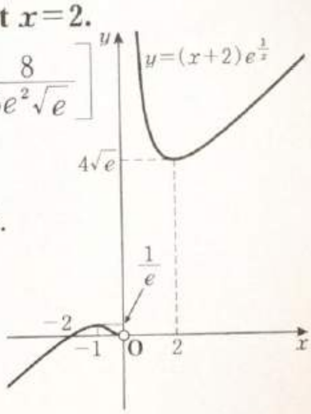
x	$\cdots$	-1	$\cdots$	$-\frac{2}{5}$	$\cdots$	0	$\cdots$	2	$\cdots$
$f'(x)$	$+$	0	$-$	$-$	$-$	/	$-$	0	$+$
$f''(x)$	$-$	$-$	$-$	0	$+$	/	$+$	$+$	$+$
$f(x)$	↗	$\frac{1}{e}$	↘	$\frac{8\sqrt{e}}{5e^3}$	↘	/	↘	$4\sqrt{e}$	↗

Therefore, **the relative maximum value is $\frac{1}{e}$, at $x=-1$** and **the relative minimum value is $4\sqrt{e}$, at $x=2$.**

The inflection point is $\left(-\frac{2}{5},\ \frac{8\sqrt{e}}{5e^3}\right)$. $\left[\frac{8\sqrt{e}}{5e^3}=\frac{8}{5e^2\sqrt{e}}\right]$

Also, $\lim_{x\to 0^+} f(x)=\infty$, $\lim_{x\to 0^-} f(x)=0$

From the above, the graph is as shown on the right.



O182b

(2) Find the limit from the right, $\lim_{x\to 0^+}\frac{3-f(x)}{1+2f(x)}$. ➡ N104

[Sol] From (1), $\lim_{x\to 0^+} f(x)=\infty$

Therefore,

$$\lim_{x\to 0^+}\frac{3-f(x)}{1+2f(x)}=\lim_{x\to 0^+}\frac{\frac{3}{f(x)}-1}{\frac{1}{f(x)}+2}=-\frac{1}{2}$$

(3) Determine if $\lim_{x\to 0}\frac{3-f(x)}{1+2f(x)}$ exists. If it exists, find the limit. If not, state the reason.

➡ N105

[Sol] From (1), $\lim_{x\to 0^-} f(x)=0$

Therefore,

$$\lim_{x\to 0^-}\frac{3-f(x)}{1+2f(x)}=3$$

Thus, from (2), $\lim_{x\to 0^+}\frac{3-f(x)}{1+2f(x)}\neq\lim_{x\to 0^-}\frac{3-f(x)}{1+2f(x)}$

Therefore, $\lim_{x\to 0}\frac{3-f(x)}{1+2f(x)}$ **does not exist.**

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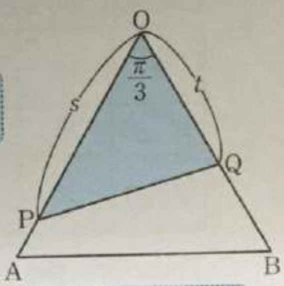
1. Place points P and Q on sides OA and OB of equilateral triangle OAB with side length 1. Given that the area of $\triangle$OPQ becomes a half of the area of $\triangle$OAB, find the range of the length of side PQ.



[Sol] Let OP$=s$ and OQ$=t$.

$$\triangle\mathrm{OAB}=\frac{1}{2}\cdot 1\cdot 1\cdot\sin\frac{\pi}{3}=\frac{\sqrt{3}}{4}$$ ← $S=\frac{1}{2}bc\sin A$

$$\triangle\mathrm{OPQ}=\frac{1}{2}st\sin\frac{\pi}{3}=\frac{\sqrt{3}}{4}st$$

When $\triangle\mathrm{OPQ}=\frac{1}{2}\triangle\mathrm{OAB}$, $\frac{\sqrt{3}}{4}st=\frac{\sqrt{3}}{8}$

$$\therefore\ t=\frac{1}{2s}$$

Since $t\leqslant 1$, $\frac{1}{2}\leqslant s\leqslant 1$ ← As with $t\leqslant 1$, the maximum value of s is 1. Since $t=\frac{1}{2s}$ and $t\leqslant 1$, the minimum value of s is $\frac{1}{2}$ at $t=1$.

Let PQ$^2=f(s)$. From the Cosine Rule, ← $a^2=b^2+c^2-2bc\cos A$

$$f(s)=s^2+t^2-2st\cos\frac{\pi}{3}=s^2+\frac{1}{4s^2}-\frac{1}{2}$$ ← Substituting $t=\frac{1}{2s}$

$$f'(s)=2s-\frac{1}{2s^3}=\frac{(2s^2+1)(2s^2-1)}{2s^3}$$

When $f'(s)=0$ in $\frac{1}{2}<s<1$, $s=\frac{\sqrt{2}}{2}$

s	$\frac{1}{2}$	$\cdots$	$\frac{\sqrt{2}}{2}$	$\cdots$	1
$f'(s)$		$-$	0	$+$	
$f(s)$	$\frac{3}{4}$	↘	$\frac{1}{2}$	↗	$\frac{3}{4}$

From the variation table, $\frac{1}{2}\leqslant f(s)\leqslant\frac{3}{4}$

$$\therefore\ \frac{\sqrt{2}}{2}\leqslant \mathrm{PQ}\leqslant\frac{\sqrt{3}}{2}$$ ← PQ$^2=f(s)$

O183b

2. Given function $f(x)=\dfrac{-3x+7}{x^2-2x+2}$, solve the following questions.

(1) Find the relative extreme values of $f(x)$ and the corresponding values of x.

➡ O13

[Sol] $f'(x)=\dfrac{-3(x^2-2x+2)-(-3x+7)(2x-2)}{(x^2-2x+2)^2}=\dfrac{(3x-2)(x-4)}{(x^2-2x+2)^2}$

When $f'(x)=0$, $x=\dfrac{2}{3}$, 4

Creating the variation table,

x	$\cdots$	$\frac{2}{3}$	$\cdots$	4	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	$\frac{9}{2}$	$\searrow$	$-\frac{1}{2}$	$\nearrow$

Therefore, **the relative maximum value is $\frac{9}{2}$, at $x=\frac{2}{3}$** and

the relative minimum value is $-\frac{1}{2}$, at $x=4$.

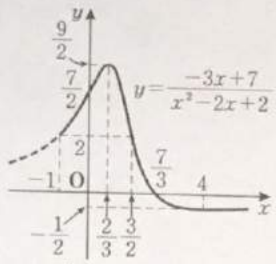
(2) Let a be a positive constant. In the domain $-1 \leqslant x \leqslant a$, find the maximum and minimum values of $f(x)$ and the corresponding values of x.

[Sol] Solving $f(x)=f(-1)$, i.e. $\dfrac{-3x+7}{x^2-2x+2}=2$,

then $x=-1$, $\dfrac{3}{2}$ ⟵ Find x satisfying $f(x)=f(-1)$

Also, since $\lim\limits_{x\to\infty} f(x)=0$,

the graph of $y=f(x)$ is as shown on the right. Therefore,



when $0<a<\frac{2}{3}$, the maximum value is $f(a)=\dfrac{-3a+7}{a^2-2a+2}$ and

the minimum value is $f(-1)=2$

when $\frac{2}{3}\leqslant a<\frac{3}{2}$, the maximum value is $f\left(\frac{2}{3}\right)=\frac{9}{2}$ and

the minimum value is $f(-1)=2$

when $a=\frac{3}{2}$, the maximum value is $f\left(\frac{2}{3}\right)=\frac{9}{2}$ and

the minimum value is $f(-1)=f\left(\frac{3}{2}\right)=2$

when $\frac{3}{2}<a<4$, the maximum value is $f\left(\frac{2}{3}\right)=\frac{9}{2}$ and

the minimum value is $f(a)=\dfrac{-3a+7}{a^2-2a+2}$

when $4\leqslant a$, the maximum value is $f\left(\frac{2}{3}\right)=\frac{9}{2}$ and

the minimum value is $f(4)=-\frac{1}{2}$.

O184a

KUMON

O 184

Applications of Calculus 1

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1. Given function $f(x)=\dfrac{\ln x}{x}$ $(x>0)$, solve the following questions.

(1) Differentiate the function $y=f(x)$. ➡ N173

[Sol] $y'=\dfrac{\frac{1}{x}\cdot x-\ln x\cdot 1}{x^2}=\dfrac{1-\ln x}{x^2}$

(2) Find the relative extreme values of the function $y=f(x)$. ➡ O13

[Sol] From (1), when $y'=0$, $1-\ln x=0$, i.e. $x=e$

Creating the variation table,

x	0	$\cdots$	e	$\cdots$
y'	/	$+$	0	$-$
y	/	$\nearrow$	$\dfrac{1}{e}$	$\searrow$

Therefore, **the relative maximum value is $\dfrac{1}{e}$, at $x=e$** and **there are no relative minimum values.**

(3) In terms of the base of the natural logarithm e and circumference ratio π, compare $f(e)$ and $f(\pi)$ in size.

[Sol] Since $e<\pi$, from (2), $f(e)>f(\pi)$

(4) Compare e^π and π^e in size.

[Sol] Since $f(e)=\dfrac{\ln e}{e}$ and $f(\pi)=\dfrac{\ln\pi}{\pi}$, from (3),

$\dfrac{\ln e}{e}>\dfrac{\ln\pi}{\pi}$

$\therefore\ \pi\ln e>e\ln\pi$ — Multiplying both sides by πe

$\therefore\ \ln e^\pi>\ln\pi^e$ — $n\ln M=\ln M^n$

$\therefore\ e^\pi>\pi^e$ — When $a>1$, then $p<q \Leftrightarrow \log_a p<\log_a q$ (L19)

O184b

2. Find the number of tangents to curve $y=xe^{-x}$ passing through point $(0, b)$.
   It is possible to use $\lim_{x\to\infty} x^n e^{-x}=0$, where n is a natural number. ➡ O42

[Sol] Let $f(x)=xe^{-x}$.

$$f'(x)=1\cdot e^{-x}-xe^{-x}=(1-x)e^{-x}$$

Let the coordinates of the tangent point be (a, ae^{-a}). The equation of the tangent is $y-ae^{-a}=(1-a)e^{-a}(x-a)$

So, $y=(1-a)e^{-a}x+a^2e^{-a}$

Since this line passes through $(0, b)$, $b=a^2e^{-a}$

Let $g(a)=a^2e^{-a}$.

$$g'(a)=2ae^{-a}-a^2e^{-a}=a(2-a)e^{-a}$$

When $g'(a)=0$, $a=0,\ 2$

a	$\cdots$	0	$\cdots$	2	$\cdots$
$g'(a)$	$-$	0	$+$	0	$-$
$g(a)$	$\searrow$	0	$\nearrow$	$\frac{4}{e^2}$	$\searrow$

Also, $\lim_{a\to\infty} g(a)=0$, $\lim_{a\to-\infty} g(a)=\infty$ ⟵ Substituting $x=a$ and $n=2$ into $\lim_{x\to\infty} x^n e^{-x}=0$, $\lim_{a\to\infty} a^2e^{-a}=0$

Therefore, the graph of $y=g(a)$ is as shown below.

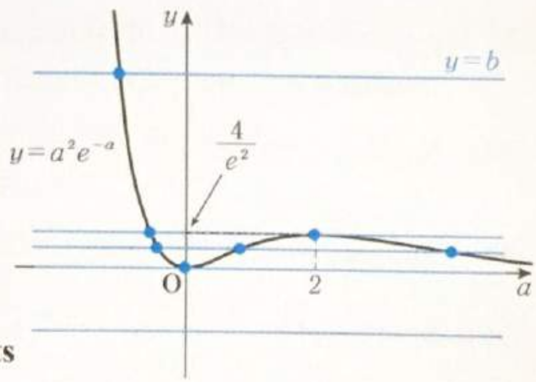
From the common points of this graph and line $y=b$, the number of tangents is:

When $0<b<\frac{4}{e^2}$, 3 tangents

When $b=\frac{4}{e^2}$, 2 tangents

When $b=0$, $b>\frac{4}{e^2}$, 1 tangent

When $b<0$, no tangents



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Let $f_0(x)=e^x$. For $n=1, 2, \ldots$, define $f_n(x)$ by $f_n(x)=xf_{n-1}'(x)$ and let $P_n(x)=e^{-x}f_n(x)$. Solve the following questions.

➡ N54

(1) Prove that $P_n(x)$ is a n^{th} degree polynomial.

[Sol] Let the statement "$P_n(x)$ is a n^{th} degree polynomial" be ①.

(i) When $n=1$,

$$P_1(x)=e^{-x}f_1(x)=xe^{-x}f_0'(x)=x$$

From the condition, $f_n(x)=xf_{n-1}'(x)$

Therefore, ① is true when $n=1$.

(ii) If ① is true when $n=k$,

using a_k and b_k,

$$P_k(x)=a_kx^k+b_kx^{k-1}+\cdots \quad (a_k\neq 0) \quad \cdots②$$

$a_kx^k+b_kx^{k-1}+\cdots$ is a k^{th} degree polynomial.

Then, since $P_n(x)=e^{-x}f_n(x)$, $f_n(x)=e^xP_n(x)$

Also, $f_n(x)=xf_{n-1}'(x)$

$$=x[e^xP_{n-1}(x)+e^xP_{n-1}'(x)]$$

$$=xe^x[P_{n-1}(x)+P_{n-1}'(x)]$$

$$\therefore\ P_n(x)=x[P_{n-1}(x)+P_{n-1}'(x)]$$

$f_{n-1}'(x)$
$=[e^xP_{n-1}(x)]'$
$=(e^x)'P_{n-1}(x)+e^xP_{n-1}'(x)$

Therefore, when $n=k+1$, rearranging by using ②,

$$P_{k+1}(x)=x[P_k(x)+P_k'(x)]$$

$$=x\{(a_kx^k+b_kx^{k-1}+\cdots)+[ka_kx^{k-1}+(k-1)b_kx^{k-2}+\cdots]\}$$

$$=a_kx^{k+1}+(ka_k+b_k)x^k+\cdots$$

Since $a_k\neq 0$, $P_{k+1}(x)$ is a $(k+1)^{\text{th}}$ degree polynomial.

Thus, ① is also true when $n=k+1$.

From (i) and (ii), ① is true for all natural numbers n.

Monomials are also treated as polynomials in this question.

O185b

(2) For $P_n(x)$, let the coefficient of x^n be a_n and the coefficient of x^{n-1} be b_n. Find a_n and b_n.

[Sol] From (1), $a_{k+1}=a_k$ $\cdots③$

$b_{k+1}=ka_k+b_k$ $\cdots④$

From (ii) in (1),
$P_k(x)=a_k x^k+b_k x^{k-1}+\cdots$
$P_{k+1}(x)=a_k x^{k+1}+(ka_k+b_k)x^k+\cdots$

Also, since $P_1(x)=x$, $a_1=1$, $b_1=0$

From ③, $a_n=1$

Since the 1st term of the sequence $\{a_n\}$ is 1, from ③, $a_1=a_2=\cdots=1$ (a constant)

From ④, $b_{n+1}-b_n=na_n=n$

Since the general term of the sequence of differences of $\{b_n\}$ is n,

when $n \geqq 2$,

$$b_n=b_1+\sum_{k=1}^{n-1}k$$

Sequence of Differences and General Term (N32)

$$=\frac{1}{2}(n-1)n \quad \cdots⑤$$

Since $b_1=0$, ⑤ is also true when $n=1$.

$$\therefore\ a_n=1,\ b_n=\frac{1}{2}(n-1)n$$

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1. Solve the following questions.

(1) Find the range of values of x for which the infinite geometric series
$$1-(x^2+x-3)+(x^2+x-3)^2-(x^2+x-3)^3+\cdots$$
converges. Then, find the sum. ➡ N83

The common ratio r is $|r|<1$.

[Sol] Since the common ratio is $-(x^2+x-3)$, $-1<-(x^2+x-3)<1$

So, $-1<-(x^2+x-3)$ …① and also $-(x^2+x-3)<1$ …②

From ①, $x^2+x-4<0$; therefore, $\dfrac{-1-\sqrt{17}}{2}<x<\dfrac{-1+\sqrt{17}}{2}$ …③

From ②, $(x+2)(x-1)>0$; therefore, $x<-2,\ 1<x$ …④

From ③ and ④, $\dfrac{-1-\sqrt{17}}{2}<x<-2,\ 1<x<\dfrac{-1+\sqrt{17}}{2}$

Also, the sum is $\dfrac{1}{1-[-(x^2+x-3)]}=\dfrac{1}{x^2+x-2}$ $\quad S=\dfrac{a}{1-r}$

(2) Let the sum of the series from (1) be $f(x)$. Find $\int f(x)dx$. ➡ O57

[Sol] Since $f(x)=\dfrac{1}{x^2+x-2}=\dfrac{1}{(x-1)(x+2)}$,

let $\dfrac{1}{(x-1)(x+2)}=\dfrac{a}{x-1}+\dfrac{b}{x+2}$.

$$1=a(x+2)+b(x-1)$$
$$=(a+b)x+(2a-b)$$
$$\therefore \begin{cases} a+b=0 \\ 2a-b=1 \end{cases}$$
$$\therefore\ a=\frac{1}{3},\ b=-\frac{1}{3}$$

Therefore,
$$\int f(x)dx=\int \frac{1}{3}\left(\frac{1}{x-1}-\frac{1}{x+2}\right)dx=\frac{1}{3}(\ln|x-1|-\ln|x+2|)+C$$
$$=\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|+C$$

From (1), $\dfrac{x-1}{x+2}>0$; therefore, $\int f(x)dx=\dfrac{1}{3}\ln\dfrac{x-1}{x+2}+C$

0186b

2. For the natural number n, let $I(n)=\int_0^1 x^n e^{-x^2}dx$.

(1) Prove the equation $I(n+2)=-\frac{1}{2}e^{-1}+\frac{n+1}{2}I(n)$.

[Sol] $I(n+2)=\int_0^1 x^{n+2}e^{-x^2}dx$

$$=\int_0^1 x^{n+1}\cdot xe^{-x^2}dx$$

$$=\left[-\frac{1}{2}x^{n+1}e^{-x^2}\right]_0^1+\frac{n+1}{2}\int_0^1 x^n e^{-x^2}dx$$

$$=-\frac{1}{2}e^{-1}+\frac{n+1}{2}I(n)$$

$\int x^{n+1}\cdot xe^{-x^2}dx=\int x^{n+1}\left(-\frac{1}{2}e^{-x^2}\right)'dx$
$=-\frac{1}{2}x^{n+1}e^{-x^2}-\int (x^{n+1})'\left(-\frac{1}{2}e^{-x^2}\right)dx$

(2) Prove the inequality $0\leqslant I(n)\leqslant\frac{1}{n+1}$.

➡ 0124

[Sol] Since $0\leqslant x^n e^{-x^2}\leqslant x^n$ when $0\leqslant x\leqslant 1$,

When $0\leqslant x\leqslant 1$,
$0<e^{-1}\leqslant e^{-x^2}\leqslant 1$,
$x^n\geqslant 0$

$$0\leqslant\int_0^1 x^n e^{-x^2}dx\leqslant\int_0^1 x^n dx$$

$$\therefore\ 0\leqslant I(n)\leqslant\frac{1}{n+1}$$

$\int_0^1 x^n dx=\left[\frac{1}{n+1}x^{n+1}\right]_0^1=\frac{1}{n+1}$

(3) Find $\lim_{n\to\infty} nI(n)$.

[Sol] Since $\lim_{n\to\infty}\frac{1}{n+1}=0$, from (2), $\lim_{n\to\infty}I(n)=0$

Limits of Sequences and Their Relationships (N67)

Also, from (1), $I(n)=\frac{2}{n+1}\left[I(n+2)+\frac{1}{2e}\right]$

Therefore,

$$\lim_{n\to\infty} nI(n)=\lim_{n\to\infty}\frac{2n}{n+1}\left[I(n+2)+\frac{1}{2e}\right]$$

$$=\lim_{n\to\infty}\frac{2}{1+\frac{1}{n}}\left[I(n+2)+\frac{1}{2e}\right]$$

$$=\frac{1}{e}$$

$\lim_{n\to\infty}I(n+2)=\lim_{n\to\infty}I(n)=0$

Applications of Calculus 1

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1. Given $f(x)=e^x$ and $g_n(x)=ne^{-x}$ (n is a natural number greater than or equal to 2), solve the following questions.

(1) Find the area S_n enclosed by $y=f(x)$, $y=g_n(x)$ and the y-axis.

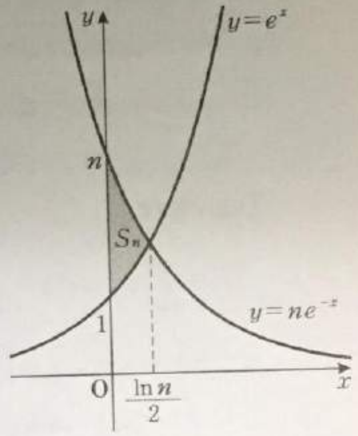
➡ O132

[Sol] When $e^x=ne^{-x}$,

$$e^{2x}=n$$

$$\therefore\ 2x=\ln n$$

$$\therefore\ x=\frac{\ln n}{2}$$



Since $e^x \leqq ne^{-x}$ in $0 \leqq x \leqq \frac{\ln n}{2}$,

$$S_n=\int_0^{\frac{\ln n}{2}}(ne^{-x}-e^x)dx$$

$$=\Big[-ne^{-x}-e^x\Big]_0^{\frac{\ln n}{2}}$$

$$=n-2\sqrt{n}+1 \quad [=(\sqrt{n}-1)^2]$$

(2) Find $\lim_{n\to\infty}(S_{n+1}-S_n)$.

➡ N65

[Sol] From (1),

$$\lim_{n\to\infty}(S_{n+1}-S_n)=\lim_{n\to\infty}\{[(n+1)-2\sqrt{n+1}+1]-(n-2\sqrt{n}+1)\}$$

$$=\lim_{n\to\infty}[1-2(\sqrt{n+1}-\sqrt{n})]$$

$$=\lim_{n\to\infty}\left[1-\frac{2(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}\right]$$

$$=\lim_{n\to\infty}\left(1-\frac{2}{\sqrt{n+1}+\sqrt{n}}\right)$$

$$=1$$

Considering $\sqrt{n+1}-\sqrt{n}=\frac{\sqrt{n+1}-\sqrt{n}}{1}$, then multiplying the numerator and the denominator by $\sqrt{n+1}+\sqrt{n}$

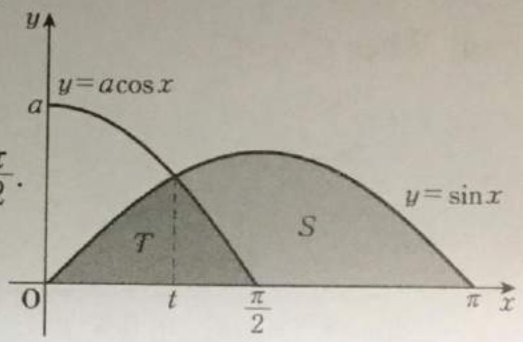
0187b

2. On a coordinate plane, let S be the area enclosed by curve $y=\sin x$ $(0 \leqslant x \leqslant \pi)$ and the x-axis, and let T be the area enclosed by curves $y=\sin x$ $\left(0 \leqslant x \leqslant \frac{\pi}{2}\right)$ and $y=a\cos x$ $\left(0 \leqslant x \leqslant \frac{\pi}{2}\right)$, and the x-axis. Given these conditions, find the value of a such that $S:T=3:1$. (a is a positive real number.)

➡ 0139

[Sol] $S=\int_0^{\pi} \sin x\,dx=\left[-\cos x\right]_0^{\pi}=2$ ⋯①

Also, let t be the x-coordinate of the point of intersection of two curves $y=\sin x$ and $y=a\cos x$ in $0 \leqslant x \leqslant \frac{\pi}{2}$.



$\sin t=a\cos t$ ⋯②

Therefore,

$$T=\int_0^{t} \sin x\,dx+\int_t^{\frac{\pi}{2}} a\cos x\,dx$$

$$=\left[-\cos x\right]_0^{t}+\left[a\sin x\right]_t^{\frac{\pi}{2}}$$

$$=-\cos t-a\sin t+a+1 \quad \cdots③$$

Since $S:T=3:1$, $S=3T$; therefore, from ① and ③,

$$2=3(-\cos t-a\sin t+a+1)$$

$$\therefore\ 3\cos t+3a\sin t-3a-1=0 \quad \cdots④$$

Substituting $\sin t=a\cos t$ and then $\cos t=\frac{\sin t}{a}$ into ④

From ② and ④, $\cos t=\frac{3a+1}{3(a^2+1)}$, $\sin t=\frac{a(3a+1)}{3(a^2+1)}$

$\cos^2 t+\sin^2 t=1$

$$\therefore\ \left[\frac{3a+1}{3(a^2+1)}\right]^2+\left[\frac{a(3a+1)}{3(a^2+1)}\right]^2=1$$

$$(3a+1)^2+a^2(3a+1)^2=9(a^2+1)^2$$

$$(a^2+1)(3a+1)^2=9(a^2+1)^2$$

$$(3a+1)^2=9(a^2+1)$$

$$\therefore\ \boldsymbol{a=\frac{4}{3}}$$

This satisfies $a>0$.

Applications of Calculus 1

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Let n be 0 or a positive integer, and $I_n=\int_{-\pi}^{\pi}x^n\cos x\,dx$, $J_n=\int_{-\pi}^{\pi}x^n\sin x\,dx$.

(1) When $n\geqq 1$, find the expression relating I_n with J_{n-1} and J_n with I_{n-1}.

[Sol] $I_n=\Big[x^n\sin x\Big]_{-\pi}^{\pi}-n\int_{-\pi}^{\pi}x^{n-1}\sin x\,dx$

$=-nJ_{n-1}$

$$I_n=\int_{-\pi}^{\pi}x^n(\sin x)'dx=\Big[x^n\sin x\Big]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}(x^n)'\sin x\,dx$$

$J_n=\Big[-x^n\cos x\Big]_{-\pi}^{\pi}+n\int_{-\pi}^{\pi}x^{n-1}\cos x\,dx$

$=\pi^n-(-\pi)^n+nI_{n-1}$

$$J_n=\int_{-\pi}^{\pi}x^n(-\cos x)'dx=\Big[-x^n\cos x\Big]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}(x^n)'(-\cos x)dx$$

(2) Find the values of I_n for $n=0, 1, 2, 3, 4$.

[Sol] $I_0=\int_{-\pi}^{\pi}\cos x\,dx=\Big[\sin x\Big]_{-\pi}^{\pi}=0$

$J_0=\int_{-\pi}^{\pi}\sin x\,dx=\Big[-\cos x\Big]_{-\pi}^{\pi}=0$

From (1),

$I_1=-J_0=0$

$I_2=-2J_1$

$=-2[\pi-(-\pi)+I_0]$

$=-4\pi$

$I_3=-3J_2$

$=-3[\pi^2-(-\pi)^2+2I_1]$

$=0$

$I_4=-4J_3$

$=-4[\pi^3-(-\pi)^3+3I_2]$

$=-8\pi^3+48\pi$

Alternative Solution

$I_1=\int_{-\pi}^{\pi}x\cos x\,dx$

$=\Big[x\sin x\Big]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\sin x\,dx$

$=-\Big[-\cos x\Big]_{-\pi}^{\pi}=0$

$I_2=\int_{-\pi}^{\pi}x^2\cos x\,dx$

$=\Big[x^2\sin x\Big]_{-\pi}^{\pi}-2\int_{-\pi}^{\pi}x\sin x\,dx$

$=-2\left(\Big[-x\cos x\Big]_{-\pi}^{\pi}+\int_{-\pi}^{\pi}\cos x\,dx\right)$

$=-2\left(2\pi+\Big[\sin x\Big]_{-\pi}^{\pi}\right)=-4\pi$

$I_3=\int_{-\pi}^{\pi}x^3\cos x\,dx$

$=\Big[x^3\sin x\Big]_{-\pi}^{\pi}-3\int_{-\pi}^{\pi}x^2\sin x\,dx$

$=-3\left(\Big[-x^2\cos x\Big]_{-\pi}^{\pi}+2\int_{-\pi}^{\pi}x\cos x\,dx\right)=0$

$I_4=\int_{-\pi}^{\pi}x^4\cos x\,dx$

$=\Big[x^4\sin x\Big]_{-\pi}^{\pi}-4\int_{-\pi}^{\pi}x^3\sin x\,dx$

$=-4\left(\Big[-x^3\cos x\Big]_{-\pi}^{\pi}+3\int_{-\pi}^{\pi}x^2\cos x\,dx\right)$

$=-4(2\pi^3-12\pi)=-8\pi^3+48\pi$

O188b

(3) Find the quadratic expression of x, $f(x)$, such that $\int_{-\pi}^{\pi} x^n f(x)\cos x\,dx = 4\pi$ for all $n=0, 1, 2$.

[Sol] Let $f(x)=ax^2+bx+c\ (a\neq 0)$.

When $n=0$,

$$\int_{-\pi}^{\pi}(ax^2+bx+c)\cos x\,dx=4\pi$$

$$a\int_{-\pi}^{\pi}x^2\cos x\,dx+b\int_{-\pi}^{\pi}x\cos x\,dx+c\int_{-\pi}^{\pi}\cos x\,dx=4\pi$$

$\therefore\ aI_2+bI_1+cI_0=4\pi$

Substituting $I_0=0, I_1=0, I_2=-4\pi$

From (2), since $-4\pi a=4\pi$, $a=-1$ $\cdots$①

This satisfies $a\neq 0$.

When $n=1$,

$$\int_{-\pi}^{\pi}x(ax^2+bx+c)\cos x\,dx=4\pi$$

From ①, $-\int_{-\pi}^{\pi}x^3\cos x\,dx+b\int_{-\pi}^{\pi}x^2\cos x\,dx+c\int_{-\pi}^{\pi}x\cos x\,dx=4\pi$

$\therefore\ -I_3+bI_2+cI_1=4\pi$

Substituting $I_1=0, I_2=-4\pi, I_3=0$

From (2), $-4\pi b=4\pi$; therefore, $b=-1$ $\cdots$②

When $n=2$,

$$\int_{-\pi}^{\pi}x^2(ax^2+bx+c)\cos x\,dx=4\pi$$

From ① and ②, $-\int_{-\pi}^{\pi}x^4\cos x\,dx-\int_{-\pi}^{\pi}x^3\cos x\,dx+c\int_{-\pi}^{\pi}x^2\cos x\,dx=4\pi$

$\therefore\ -I_4-I_3+cI_2=4\pi$

Substituting $I_2=-4\pi, I_3=0, I_4=-8\pi^3+48\pi$

From (2), $-(-8\pi^3+48\pi)-4\pi c=4\pi$; therefore, $c=2\pi^2-13$ $\cdots$③

From ①~③, $f(x)=-x^2-x+2\pi^2-13$

Applications of Calculus 1

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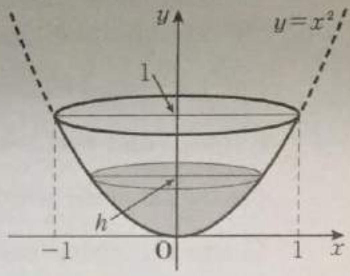
1. The container formed by rotating curve $y=x^2$ $(0 \leqslant x \leqslant 1)$ once about the y-axis is filled with water. This container has a drain at the bottom. The water will be drained from time $t=0$. Let the height and the amount of water remaining in the container at time t be h and V respectively. The rate of change of V over time, $\dfrac{dV}{dt}$, is given by $\dfrac{dV}{dt}=-\sqrt{h}$.

(1) Express the rate of change of the height h of water, $\dfrac{dh}{dt}$, in terms of h.



[Sol] $V=\pi\displaystyle\int_0^h y\,dy=\pi\left[\frac{1}{2}y^2\right]_0^h=\frac{1}{2}\pi h^2$

Differentiating both sides with respect to t,

$$\frac{dV}{dt}=\pi h\cdot\frac{dh}{dt} \quad \cdots ① \longleftarrow \text{※}$$

Since $\dfrac{dV}{dt}=-\sqrt{h}$, from ①,

$$\frac{dh}{dt}=-\frac{\sqrt{h}}{\pi h} \quad \left[=-\frac{1}{\pi\sqrt{h}}\right]$$

※ $\dfrac{d}{dt}\left(\dfrac{1}{2}\pi h^2\right)=\dfrac{d}{dh}\left(\dfrac{1}{2}\pi h^2\right)\cdot\dfrac{dh}{dt}=\pi h\cdot\dfrac{dh}{dt}$

(2) Find the time T necessary for all the water to be drained from this container.

[Sol] From (1), since $\dfrac{dt}{dh}=-\pi\sqrt{h}$, $\longleftarrow$ $\dfrac{dy}{dx}=\dfrac{1}{\frac{dx}{dy}}$

$$T=\int_1^0(-\pi\sqrt{h})\,dh=\pi\int_0^1\sqrt{h}\,dh=\pi\left[\frac{2}{3}h^{\frac{3}{2}}\right]_0^1=\frac{2}{3}\pi \longleftarrow$$

The time T is the same as the change in the time t between $h=1$ and $h=0$.

Alternative Solution

From (1), $\sqrt{h}\dfrac{dh}{dt}=-\dfrac{1}{\pi}$

Integrating both sides with respect to t, $\displaystyle\int\sqrt{h}\,dh=\int\left(-\frac{1}{\pi}\right)dt$

$\therefore\ \dfrac{2}{3}h\sqrt{h}=-\dfrac{1}{\pi}t+C$

When $t=0$, $h=1$; therefore, $C=\dfrac{2}{3}$

When $h=0$, $t=T$; therefore, $\dfrac{1}{\pi}T=\dfrac{2}{3}$

$\therefore\ T=\dfrac{2}{3}\pi$

O189b

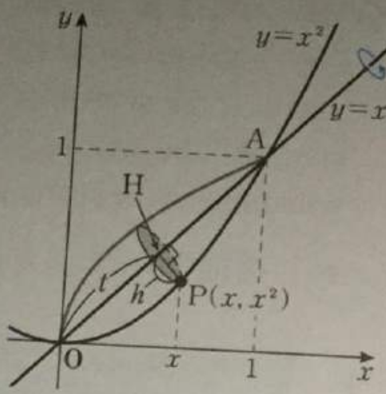
2. On the xy-plane, given a solid formed by rotating the area enclosed by parabola $y=x^2$ and line $y=x$ once about line $y=x$, find its volume. ➡ O142

[Sol] When $x^2=x$,

$x(x-1)=0$

$\therefore\ x=0,\ 1$

Finding the x-coordinate of the point of intersection of $y=x^2$ and $y=x$



Let the points of intersection be $O(0, 0)$ and $A(1, 1)$.

$OA=\sqrt{2}$

Let $0 \leqslant x \leqslant 1$. Then, drop a perpendicular PH from point $P(x, x^2)$ on the parabola to line $y=x$, and let $PH=h$ and $OH=t$.

Let the cross-sectional area of the solid cut by the plane which passes through H and is perpendicular to $y=x$ be $S(t)$ and the volume of this solid be V.

$$V=\int_0^{\sqrt{2}} S(t)dt=\pi\int_0^{\sqrt{2}} h^2 dt$$

Then, $h=\dfrac{|x-x^2|}{\sqrt{2}}=\dfrac{x-x^2}{\sqrt{2}}$

$t=\sqrt{2}x-h=\dfrac{x+x^2}{\sqrt{2}}$

$\therefore\ dt=\dfrac{1+2x}{\sqrt{2}}dx$

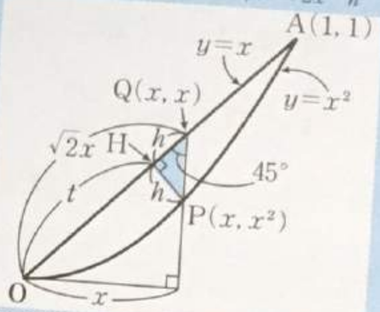
t	$0 \longrightarrow \sqrt{2}$
x	$0 \longrightarrow 1$

The length of the perpendicular dropped from point (x_1, y_1) to line $ax+by+c=0$ is $\dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

When $0 \leqslant x \leqslant 1$, $x \geqslant x^2$; therefore, $|x-x^2|=x-x^2$

Or, from the diagram below, since $\triangle$HPQ is an isosceles right-angled triangle, $h=\dfrac{PQ}{\sqrt{2}}=\dfrac{x-x^2}{\sqrt{2}}$

Also, from the diagram below, $t=\sqrt{2}x-h$



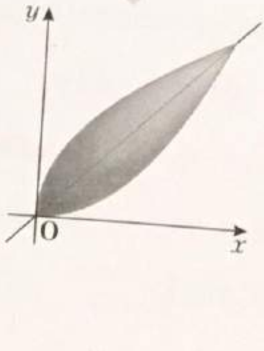
Therefore,

$$V=\pi\int_0^1 \frac{(x-x^2)^2}{2}\cdot\frac{1+2x}{\sqrt{2}}dx$$

$$=\frac{\sqrt{2}}{4}\pi\int_0^1 (x^2-3x^4+2x^5)dx$$

$$=\frac{\sqrt{2}}{4}\pi\left[\frac{1}{3}x^3-\frac{3}{5}x^5+\frac{1}{3}x^6\right]_0^1$$

$$=\frac{\sqrt{2}}{60}\pi$$



Applications of Calculus 1

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For $n=0, 1, 2, \ldots$, let $I_n=\dfrac{(-1)^n}{n!}\displaystyle\int_0^2 x^n e^x dx$. $(0!=1)$

(1) Find the value of I_0, then find the expression relating I_n with I_{n-1} when $n=1, 2, \ldots$. Then, using these, find the value of I_3.

[Sol] $I_0=\dfrac{(-1)^0}{0!}\displaystyle\int_0^2 e^x dx=\Big[e^x\Big]_0^2=e^2-1$

$$I_n=\frac{(-1)^n}{n!}\left(\Big[x^n e^x\Big]_0^2-n\int_0^2 x^{n-1}e^x dx\right)$$

$$\int x^n e^x dx=\int x^n (e^x)' dx=x^n e^x-\int (x^n)' e^x dx$$

$$=\frac{(-1)^n 2^n e^2}{n!}+\frac{(-1)^{n-1}}{(n-1)!}\int_0^2 x^{n-1}e^x dx$$

$$\frac{(-1)^n}{n!}\cdot(-n)=\frac{(-1)^{n-1}\cdot(-1)^2\cdot n}{n\cdot(n-1)!}=\frac{(-1)^{n-1}}{(n-1)!}$$

$$=\frac{(-1)^n 2^n e^2}{n!}+I_{n-1}$$

$$I_3=\frac{(-1)^3\cdot 2^3\cdot e^2}{3!}+I_2=-\frac{4}{3}e^2+\frac{(-1)^2\cdot 2^2\cdot e^2}{2!}+I_1$$

$$=\frac{2}{3}e^2+\frac{(-1)^1\cdot 2^1\cdot e^2}{1!}+I_0=-\frac{4}{3}e^2+(e^2-1) \quad \longleftarrow \boxed{I_0=e^2-1}$$

$$=-\frac{1}{3}e^2-1$$

(2) Prove the following inequality, using $e^x \leqq e^2$ when $0 \leqq x \leqq 2$.

$$\frac{1}{n!}\int_0^2 x^n e^x dx \leqq 2e^2\left(\frac{2}{3}\right)^{n-1} \quad (n=1, 2, \ldots)$$

[Sol] $\dfrac{1}{n!}\displaystyle\int_0^2 x^n e^x dx \leqq \dfrac{1}{n!}\displaystyle\int_0^2 x^n e^2 dx=\dfrac{e^2}{n!}\left[\dfrac{1}{n+1}x^{n+1}\right]_0^2=\dfrac{2^{n+1}}{(n+1)!}e^2$

Then, when $n \geqq 2$,

$$\frac{2^{n+1}}{(n+1)!}=\frac{2^2}{1\cdot 2}\cdot\frac{2^{n-1}}{3\cdot 4\cdot 5\cdot\cdots\cdot(n+1)} \leqq 2\cdot\frac{2^{n-1}}{3\cdot 3\cdot 3\cdot\cdots\cdot 3}=2\cdot\left(\frac{2}{3}\right)^{n-1} \quad \cdots ①$$

When $n=1$, $\dfrac{2^2}{2!}=2\cdot\left(\dfrac{2}{3}\right)^0$; therefore, ① is also true when $n=1$.

$$\therefore \frac{1}{n!}\int_0^2 x^n e^x dx \leqq 2e^2\left(\frac{2}{3}\right)^{n-1} \quad (n=1, 2, \ldots)$$

$n!=n(n-1)(n-2)\cdot\cdots\cdot 3\cdot 2\cdot 1$

0190b

(3) Find the limit value $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(-1)^k 2^k}{k!}$.

[Sol] From (1), $I_n - I_{n-1} = \frac{(-1)^n 2^n e^2}{n!}$; therefore,

$$\frac{(-1)^n 2^n}{n!} = \frac{1}{e^2}(I_n - I_{n-1}) \quad (n \geqq 1)$$

Thus,

$$\sum_{k=1}^{n}\frac{(-1)^k 2^k}{k!} = \sum_{k=1}^{n}\frac{1}{e^2}(I_k - I_{k-1})$$

Since I_{-1} is not defined, determine $\sum_{k=1}^{n}\frac{(-1)^k 2^k}{k!}$ first.

$$= \frac{1}{e^2}(I_n - I_0)$$

$\sum_{k=1}^{n}(I_k - I_{k-1})$
$= (\cancel{I_1} - I_0) + (\cancel{I_2} - \cancel{I_1}) + \cdots$
$\quad + (\cancel{I_{n-1}} - \cancel{I_{n-2}}) + (I_n - \cancel{I_{n-1}})$
$= I_n - I_0$

$$= \frac{1}{e^2}(I_n - e^2 + 1)$$

Therefore,

$\sum_{k=0}^{n}\frac{(-1)^k 2^k}{k!} = \frac{(-1)^0 \cdot 2^0}{0!} + \sum_{k=1}^{n}\frac{(-1)^k 2^k}{k!}$

$$\sum_{k=0}^{n}\frac{(-1)^k 2^k}{k!} = \frac{(-1)^0 \cdot 2^0}{0!} + \frac{1}{e^2}(I_n - e^2 + 1)$$

$$= \frac{1}{e^2}(I_n + 1)$$

$\frac{(-1)^0 \cdot 2^0}{0!} = 1$

From (2), $-2e^2\left(\frac{2}{3}\right)^{n-1} \leqq I_n \leqq 2e^2\left(\frac{2}{3}\right)^{n-1}$

$|I_n| = \left|\frac{(-1)^n}{n!}\int_0^2 x^n e^x dx\right|$
$= \frac{1}{n!}\int_0^2 x^n e^x dx \leqq 2e^2\left(\frac{2}{3}\right)^{n-1}$

Then, $\lim_{n\to\infty}\left[-2e^2\left(\frac{2}{3}\right)^{n-1}\right] = 0$, $\lim_{n\to\infty} 2e^2\left(\frac{2}{3}\right)^{n-1} = 0$; therefore,

$$\lim_{n\to\infty} I_n = 0$$

Limits of Sequences and Their Relationships (N67)

Thus,

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(-1)^k 2^k}{k!} = \lim_{n\to\infty}\frac{1}{e^2}(I_n + 1) = \frac{1}{e^2}$$

O191a

KUMON

Applications of Calculus 2

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1. Given function $f(x)=x^2e^{-\frac{x}{a}}\ (a>0)$, solve the following questions.

(1) When $f(x)$ has a relative maximum value at $x=c$, express c in terms of a.

➡ O13

[Sol] $f'(x)=2xe^{-\frac{x}{a}}+x^2e^{-\frac{x}{a}}\cdot\left(-\frac{1}{a}\right)=\frac{(2a-x)xe^{-\frac{x}{a}}}{a}$

When $f'(x)=0$, $x=0,\ 2a$

x	$\cdots$	0	$\cdots$	$2a$	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	Relative minimum	$\nearrow$	Relative maximum	$\searrow$

$f(x)=0$ when $x=0$, and $f(x)=\frac{4a^2}{e^2}$ when $x=2a$. However, it is only necessary to find x for which $f(x)$ has the relative maximum value here. Therefore, it is possible to write relative minimum/maximum.

From the variation table, since it has a relative maximum value at $x=2a$, $\boldsymbol{c=2a}$

(2) Express the definite integral $\int_0^c f(x)dx$ in terms of a.

➡ O113

[Sol] From (1),

$$\int_0^c f(x)dx=\int_0^{2a}x^2e^{-\frac{x}{a}}dx$$

※1

$$=\left[-ax^2e^{-\frac{x}{a}}\right]_0^{2a}+2a\int_0^{2a}xe^{-\frac{x}{a}}dx$$

※2

$$=-\frac{4a^3}{e^2}+2a\left(\left[-axe^{-\frac{x}{a}}\right]_0^{2a}+a\int_0^{2a}e^{-\frac{x}{a}}dx\right)$$

$$=-\frac{8a^3}{e^2}+2a^2\left[-ae^{-\frac{x}{a}}\right]_0^{2a}$$

$$=\boldsymbol{-\frac{10a^3}{e^2}+2a^3}\quad\left[=2a^3\left(1-\frac{5}{e^2}\right)\right]$$

※1 $\int x^2e^{-\frac{x}{a}}dx=\int x^2\left(-ae^{-\frac{x}{a}}\right)'dx=-ax^2e^{-\frac{x}{a}}-\int (x^2)'\left(-ae^{-\frac{x}{a}}\right)dx$

※2 $\int xe^{-\frac{x}{a}}dx=\int x\left(-ae^{-\frac{x}{a}}\right)'dx=-axe^{-\frac{x}{a}}-\int (x)'\left(-ae^{-\frac{x}{a}}\right)dx$

0191b

2. Solve the following questions.

(1) For function $f(x)=xe^{-2x}$, find the relative extreme values and the inflection point(s). ➡ 023

[Sol] $f'(x)=1\cdot e^{-2x}+xe^{-2x}\cdot(-2)=(1-2x)e^{-2x}$

$f''(x)=-2e^{-2x}+(1-2x)e^{-2x}\cdot(-2)=4(x-1)e^{-2x}$

When $f'(x)=0$, $x=\frac{1}{2}$. When $f''(x)=0$, $x=1$

Creating the variation table,

x	$\cdots$	$\frac{1}{2}$	$\cdots$	1	$\cdots$
$f'(x)$	$+$	0	$-$	$-$	$-$
$f''(x)$	$-$	$-$	$-$	0	$+$
$f(x)$	↗	$\frac{1}{2e}$	↘	$\frac{1}{e^2}$	↘

Therefore,

the relative maximum value is $\frac{1}{2e}$, at $x=\frac{1}{2}$ and there are no relative minimum values.

The inflection point is $\left(1, \frac{1}{e^2}\right)$.

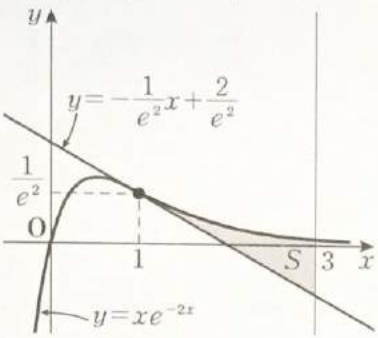
(2) Find the area enclosed by the curve $y=f(x)$, its tangent(s) at the inflection point(s) on the curve $y=f(x)$ and line $x=3$. ➡ 0135

[Sol] From (1), the inflection point is $\left(1, \frac{1}{e^2}\right)$.

Since $f'(1)=-\frac{1}{e^2}$, the equation of the tangent at point $\left(1, \frac{1}{e^2}\right)$ is

$$y-\frac{1}{e^2}=-\frac{1}{e^2}(x-1)$$

So, $y=-\frac{1}{e^2}x+\frac{2}{e^2}$



Therefore, let the area to be found be S.

$$S=\int_1^3\left[xe^{-2x}-\left(-\frac{1}{e^2}x+\frac{2}{e^2}\right)\right]dx$$

$$=\int_1^3\left(xe^{-2x}+\frac{1}{e^2}x-\frac{2}{e^2}\right)dx$$

$$=\left[-\frac{1}{2}xe^{-2x}\right]_1^3+\frac{1}{2}\int_1^3 e^{-2x}dx+\left[\frac{1}{2e^2}x^2-\frac{2}{e^2}x\right]_1^3$$

$$=-\frac{3}{2e^6}+\frac{1}{2e^2}+\frac{1}{2}\left[-\frac{1}{2}e^{-2x}\right]_1^3$$

$$=\frac{3e^4-7}{4e^6}$$

$$\int xe^{-2x}dx=\int x\left(-\frac{1}{2}e^{-2x}\right)'dx$$
$$=-\frac{1}{2}xe^{-2x}-\int(x)'\left(-\frac{1}{2}e^{-2x}\right)dx$$

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Given curve $C: y=f(x)=\sqrt{x}(\ln x-1)$, solve the following questions.

(1) Find the relative extreme values of the function $f(x)$. ➡ O13

[Sol] The domain is $x>0$.

$$f'(x)=\frac{1}{2\sqrt{x}}(\ln x-1)+\sqrt{x}\cdot\frac{1}{x}=\frac{\ln x+1}{2\sqrt{x}}$$

When $f'(x)=0$ in $x>0$, $\ln x=-1$, i.e. $x=\frac{1}{e}$

Creating the variation table,

x	0	$\cdots$	$\frac{1}{e}$	$\cdots$
$f'(x)$		$-$	0	$+$
$f(x)$		$\searrow$	$-\frac{2\sqrt{e}}{e}$	$\nearrow$

Therefore,
the relative minimum value is $-\frac{2\sqrt{e}}{e}$, at $x=\frac{1}{e}$ and there are no relative maximum values.

$\left[-\frac{2\sqrt{e}}{e}=-\frac{2}{\sqrt{e}}\right]$

(2) Find the inflection point(s) of curve C and sketch its graph. ➡ O23

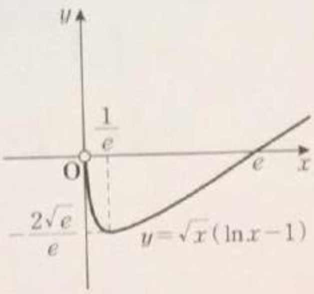
$\left(\lim_{x\to 0^+}\sqrt{x}\ln x=0\right)$

[Sol] $$f''(x)=\frac{\frac{1}{x}\cdot 2\sqrt{x}-(\ln x+1)\cdot\frac{1}{\sqrt{x}}}{4x}=\frac{1-\ln x}{4x\sqrt{x}}$$

When $f''(x)=0$ in $x>0$, $\ln x=1$, i.e. $x=e$

Creating the variation table,

x	0	$\cdots$	$\frac{1}{e}$	$\cdots$	e	$\cdots$
$f'(x)$		$-$	0	$+$	$+$	$+$
$f''(x)$		$+$	$+$	$+$	0	$-$
$f(x)$		$\searrow$	$-\frac{2\sqrt{e}}{e}$	$\nearrow$	0	$\nearrow$



Therefore, the inflection point is $(e, 0)$.

Also, since $\lim_{x\to 0^+} f(x)=0$, the graph is as shown above.

O192b

(3) Find the indefinite integral $I=\int\sqrt{x}\ln x\,dx$.

➡ O82

[Sol] $I=\int\left(\frac{2}{3}x^{\frac{3}{2}}\right)'\ln x\,dx$

$=\frac{2}{3}x^{\frac{3}{2}}\ln x-\frac{2}{3}\int x^{\frac{1}{2}}dx$

$$\int\left(\frac{2}{3}x^{\frac{3}{2}}\right)'\ln x\,dx=\frac{2}{3}x^{\frac{3}{2}}\ln x-\int\frac{2}{3}x^{\frac{3}{2}}(\ln x)'dx$$
$$=\frac{2}{3}x^{\frac{3}{2}}\ln x-\frac{2}{3}\int x^{\frac{3}{2}}\cdot\frac{1}{x}\,dx$$

$=\frac{2}{3}x\sqrt{x}\ln x-\frac{4}{9}x^{\frac{3}{2}}+A$

$=\frac{2}{3}x\sqrt{x}\ln x-\frac{4}{9}x\sqrt{x}+A$ **(A is the constant of integration)**

$\left[=\frac{2}{9}x\sqrt{x}(3\ln x-2)+A\right.$ **(A is the constant of integration)**$\left.\right]$

(4) Find the area S enclosed by curve C and curve $y=\sqrt{x}$ in $x\geqslant e$.

[Sol] When $\sqrt{x}(\ln x-1)=\sqrt{x}$,

➡ O132

$\sqrt{x}(\ln x-2)=0$

$\therefore\ \ln x=2$, i.e. $x=e^2$

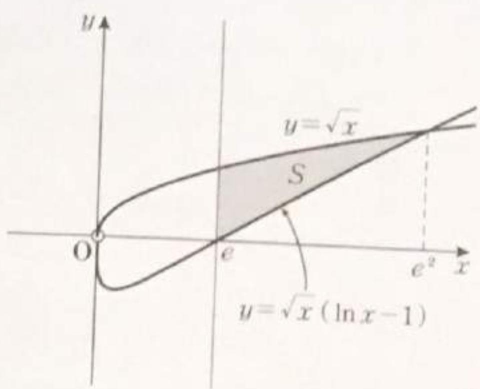
Since $\sqrt{x}(\ln x-1)\leqslant\sqrt{x}$ in $e\leqslant x\leqslant e^2$,

$S=\int_e^{e^2}[\sqrt{x}-\sqrt{x}(\ln x-1)]dx$

$=\int_e^{e^2}(2\sqrt{x}-\sqrt{x}\ln x)dx$

$=\left[\frac{4}{3}x^{\frac{3}{2}}-\left(\frac{2}{3}x^{\frac{3}{2}}\ln x-\frac{4}{9}x^{\frac{3}{2}}\right)\right]_e^{e^2}$

$=\frac{4}{9}e^3-\frac{10}{9}e\sqrt{e}$



Applications of Calculus 2

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1. Find the real number a which satisfies the following two equations, then find the functions $f(x)$ and $g(x)$. ➡ O117

$$\int_1^x f(t)dt = xg(x)+ax+3,\quad g(x)=x^2+x\int_0^1 f(t)dt+1$$

[Sol] Let $\int_0^1 f(t)dt = k$.

$$g(x)=x^2+kx+1 \quad \cdots ①$$

$$\int_1^x f(t)dt = xg(x)+ax+3 \quad \cdots ②$$

Differentiating both sides of ② with respect to x, from ①,

$$\begin{aligned} f(x) &= 1\cdot g(x)+xg'(x)+a \\ &= (x^2+kx+1)+x(2x+k)+a \\ &= 3x^2+2kx+a+1 \end{aligned}$$

$\frac{d}{dx}\int_a^x f(t)dt = f(x)$

Therefore,

$$\begin{aligned} \int_0^1 f(t)dt &= \int_0^1 (3t^2+2kt+a+1)dt \\ &= \Big[t^3+kt^2+(a+1)t\Big]_0^1 \\ &= k+a+2 \end{aligned}$$

Thus, since $k=k+a+2$, $a=-2$

$\int_0^1 f(t)dt = k$

Also, substituting $x=1$ into ②,

$$\int_1^1 f(t)dt = g(1)+a+3$$

$$0=k+3$$

$$\therefore\ k=-3$$

$\int_a^a f(x)dx=0$

From ①, $g(1)=k+2$

$$\therefore\ f(x)=3x^2-6x-1,\quad g(x)=x^2-3x+1$$

0193b

2. Let $a>1$. Given a solid formed by rotating the area enclosed by two lines $y=ax$ and $y=\frac{x}{a}$ and curve $y=\frac{1}{ax}$ in $x \geqq 0$ once about the x-axis, find the maximum value of volume $V(a)$ of this solid.

➡ 031, 0145

[Sol] When $ax=\frac{1}{ax}$, ⟵ Finding the x-coordinate of the point of intersection of $y=ax$ and $y=\frac{1}{ax}$

$a^2x^2-1=0$

$(ax+1)(ax-1)=0$

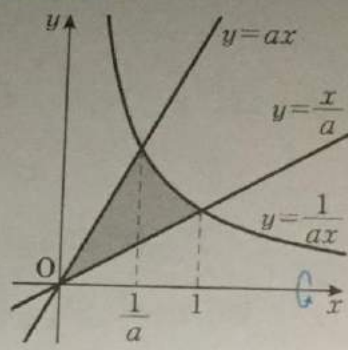
Since $a>1$ and $x \geqq 0$, $x=\frac{1}{a}$

When $\frac{x}{a}=\frac{1}{ax}$, ⟵ Finding the x-coordinate of the point of intersection of $y=\frac{x}{a}$ and $y=\frac{1}{ax}$

$x^2-1=0$

$(x+1)(x-1)=0$

Since $x \geqq 0$, $x=1$



Therefore,

$$V(a)=\pi\int_0^{\frac{1}{a}}(ax)^2dx+\pi\int_{\frac{1}{a}}^1\left(\frac{1}{ax}\right)^2dx-\pi\int_0^1\left(\frac{x}{a}\right)^2dx$$

$$=\pi a^2\int_0^{\frac{1}{a}}x^2dx+\frac{\pi}{a^2}\int_{\frac{1}{a}}^1\frac{dx}{x^2}-\frac{\pi}{a^2}\int_0^1 x^2dx$$

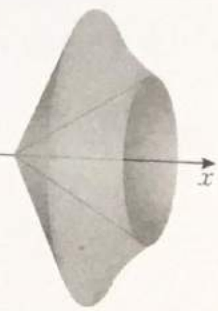
$$=\pi a^2\left[\frac{1}{3}x^3\right]_0^{\frac{1}{a}}+\frac{\pi}{a^2}\left[-\frac{1}{x}\right]_{\frac{1}{a}}^1-\frac{\pi}{a^2}\left[\frac{1}{3}x^3\right]_0^1$$

$$=\frac{4}{3}\pi\left(-\frac{1}{a^2}+\frac{1}{a}\right)$$

$$V'(a)=\frac{4}{3}\pi\left(\frac{2}{a^3}-\frac{1}{a^2}\right)=\frac{4\pi(2-a)}{3a^3}$$

When $V'(a)=0$ in $a>1$, $a=2$

a	1	$\cdots$	2	$\cdots$
$V'(a)$	/	+	0	−
$V(a)$	/	↗	$\frac{\pi}{3}$	↘



From the variation table, **the maximum value is $\frac{\pi}{3}$, at $a=2$.**

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The function $f_n(x)$ $(n=0, 1, 2, \dots)$ is defined by the following two equations.

$$\begin{cases} f_0(x)=1 \\ f_n(x)=1+\int_0^x [f_{n-1}(t)+tf_{n-1}'(t)]dt \quad (n=1, 2, 3, \dots) \cdots ① \end{cases}$$

(1) Find $f_1(x)$, $f_2(x)$ and $f_3(x)$.

[Sol] $f_1(x)=1+\int_0^x [f_0(t)+tf_0'(t)]dt=1+\int_0^x dt=1+\Big[t\Big]_0^x=\mathbf{1+x}$

$f_2(x)=1+\int_0^x [f_1(t)+tf_1'(t)]dt=1+\int_0^x [(1+t)+t]dt$

$=1+\Big[t+t^2\Big]_0^x=\mathbf{1+x+x^2}$

$f_3(x)=1+\int_0^x [f_2(t)+tf_2'(t)]dt=1+\int_0^x [(1+t+t^2)+t(1+2t)]dt$

$=1+\Big[t+t^2+t^3\Big]_0^x=\mathbf{1+x+x^2+x^3}$

(2) Assume $f_n(x)$, then prove it by mathematical induction. ➡ N57

[Sol] From (1), $f_n(x)$ is assumed to be

$f_n(x)=1+x+x^2+\cdots+x^n \cdots ②$

(i) When $n=1$, in ②,

$f_1(x)=1+x$

Therefore, ② is true when $n=1$. ⟵ It coincides with $f_1(x)=1+x$ from (1).

(ii) If ② is true when $n=k$,

$f_k(x)=1+x+x^2+\cdots+x^k \cdots ③$

When $n=k+1$, rearranging ① by using ③,

$f_{k+1}(x)=1+\int_0^x [f_k(t)+tf_k'(t)]dt$

$=1+\int_0^x [(1+t+t^2+\cdots+t^k)+t(1+2t+3t^2+\cdots+kt^{k-1})]dt$

$=1+\int_0^x [1+2t+3t^2+\cdots+(k+1)t^k]dt$

$=1+\Big[t+t^2+t^3+\cdots+t^k+t^{k+1}\Big]_0^x$

$=1+x+x^2+x^3+\cdots+x^k+x^{k+1}$

Therefore, ② is also true when $n=k+1$.

From (i) and (ii), ② is true for all natural numbers n.

0194b

(3) Given $F_n(t)=\int_0^t f_n(x)dx$, find $\lim_{n\to\infty}\int_0^1 F_n(t)dt$.

[Sol] From (2),

$$F_n(t)=\int_0^t (1+x+x^2+\cdots+x^n)dx$$
$$=\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}+\frac{x^{n+1}}{n+1}\right]_0^t$$
$$=t+\frac{t^2}{2}+\frac{t^3}{3}+\cdots+\frac{t^n}{n}+\frac{t^{n+1}}{n+1}$$

Therefore,

$$\int_0^1 F_n(t)dt=\int_0^1\left(t+\frac{t^2}{2}+\frac{t^3}{3}+\cdots+\frac{t^n}{n}+\frac{t^{n+1}}{n+1}\right)dt$$
$$=\left[\frac{t^2}{1\cdot2}+\frac{t^3}{2\cdot3}+\cdots+\frac{t^{n+1}}{n(n+1)}+\frac{t^{n+2}}{(n+1)(n+2)}\right]_0^1$$
$$=\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\cdots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}$$

N37

Let $\frac{1}{k(k+1)}=\frac{a}{k}-\frac{b}{k+1}$.

$$1=a(k+1)-bk$$
$$=(a-b)k+a$$
$$\therefore \begin{cases} a-b=0 \\ a=1 \end{cases}$$
$$\therefore\ a=1,\ b=1$$

Thus,

$$\int_0^1 F_n(t)dt=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$
$$=1-\frac{1}{n+2}$$
$$\therefore\ \lim_{n\to\infty}\int_0^1 F_n(t)dt=\lim_{n\to\infty}\left(1-\frac{1}{n+2}\right)=\mathbf{1}$$

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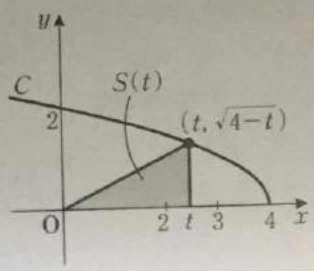
Let curve $y=\sqrt{4-x}$ be C. Given that t satisfies $2 \leqslant t \leqslant 3$, let $S(t)$ be the area of the triangle with three vertices $(t, \sqrt{4-t})$ on curve C, $(0, 0)$ and $(t, 0)$.

(1) In the interval $2 \leqslant t \leqslant 3$, find the maximum and minimum values of function $S(t)$ and the corresponding values of t.

➡ O31



[Sol] $S(t)=\frac{1}{2}t\sqrt{4-t}$

$$S'(t)=\frac{1}{2}\left(1\cdot\sqrt{4-t}+t\cdot\frac{-1}{2\sqrt{4-t}}\right)$$

$$=\frac{8-3t}{4\sqrt{4-t}}$$

When $S'(t)=0$ in $2<t<3$, $t=\frac{8}{3}$

t	2	$\cdots$	$\frac{8}{3}$	$\cdots$	3
$S'(t)$		+	0	−	
$S(t)$	$\sqrt{2}$	↗	$\frac{8\sqrt{3}}{9}$	↘	$\frac{3}{2}$

From the variation table,

the maximum value is $\frac{8\sqrt{3}}{9}$, at $t=\frac{8}{3}$ and

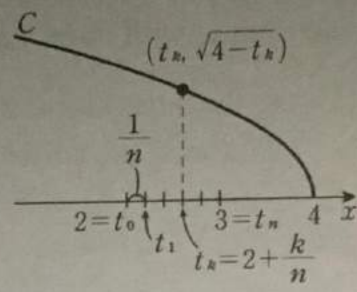
the minimum value is $\sqrt{2}$, at $t=2$.

0195b

(2) Dividing the interval [2, 3] into n equal subintervals, let both boundaries and dividing points be $t_0=2,\ t_1,\ t_2,\ \dots,\ t_{n-1},\ t_n=3$ in order. Find the limit value $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}S(t_k)$.

➡ 0102, 0122

[Sol] Since $t_k=2+\dfrac{k}{n}\quad(k=0,\ 1,\ 2,\ \dots,\ n)$,



$$S(t_k)=\frac{1}{2}\left(2+\frac{k}{n}\right)\sqrt{4-\left(2+\frac{k}{n}\right)}$$
$$=\frac{1}{2}\left(2+\frac{k}{n}\right)\sqrt{2-\frac{k}{n}}$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}S(t_k)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2}\left(2+\frac{k}{n}\right)\sqrt{2-\frac{k}{n}}$$
$$=\frac{1}{2}\int_0^1(2+x)\sqrt{2-x}\,dx$$

$$\boxed{\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)=\int_0^1 f(x)dx}$$

Let $\sqrt{2-x}=u$. Since $2-x=u^2$, $x=2-u^2$, $dx=-2u\,du$

Thus,

x	$0 \longrightarrow 1$
u	$\sqrt{2} \longrightarrow 1$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}S(t_k)=\frac{1}{2}\int_{\sqrt{2}}^{1}(4-u^2)u\cdot(-2u)du$$
$$=\int_1^{\sqrt{2}}(4u^2-u^4)du$$
$$=\left[\frac{4}{3}u^3-\frac{1}{5}u^5\right]_1^{\sqrt{2}}$$
$$=\mathbf{\frac{28\sqrt{2}-17}{15}}$$

$$\boxed{\int_b^a f(x)dx=-\int_a^b f(x)dx}$$

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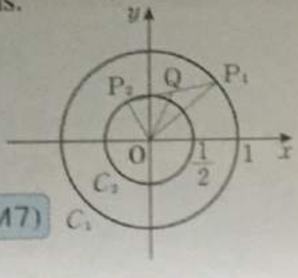
Let the circles whose centers are at the origin with radius 1 and radius $\frac{1}{2}$ be C_1 and C_2 respectively. Point $P_1(\cos\theta, \sin\theta)$ is located on C_1 and point $P_2\left(\frac{1}{2}\cos 3\theta, \frac{1}{2}\sin 3\theta\right)$ is located on C_2. $\left(0 \leqq \theta < \frac{\pi}{2}\right)$

Let the midpoint of line segment P_1P_2 be Q, and let the distance between point Q and the origin be $r(\theta)$. Then, solve the following questions.



(1) Find the range of the x-coordinate of point Q.

[Sol] Let the coordinates of point Q be (x, y).

$$x=\frac{1}{2}\left(\cos\theta+\frac{1}{2}\cos 3\theta\right) \quad \cdots①$$

$$y=\frac{1}{2}\left(\sin\theta+\frac{1}{2}\sin 3\theta\right) \quad \cdots②$$

← Midpoint (M7)

Also, $x=\frac{1}{2}\left[\cos\theta+\frac{1}{2}(4\cos^3\theta-3\cos\theta)\right]$ ← $\cos 3\theta = 4\cos^3\theta-3\cos\theta$

$$=\cos^3\theta-\frac{1}{4}\cos\theta$$

Let $\cos\theta=t$. Since $0 \leqq \theta < \frac{\pi}{2}$, $0 < t \leqq 1$

$$x=t^3-\frac{1}{4}t$$

$$\frac{dx}{dt}=3t^2-\frac{1}{4}$$

When $\frac{dx}{dt}=0$ in $0 < t < 1$, $t=\frac{\sqrt{3}}{6}$

t	0	$\cdots$	$\frac{\sqrt{3}}{6}$	$\cdots$	1
$\frac{dx}{dt}$	/	$-$	0	$+$	
x	/	↘	$-\frac{\sqrt{3}}{36}$	↗	$\frac{3}{4}$

Also, $\lim\limits_{t\to 0^+} x=\lim\limits_{t\to 0^+}\left(t^3-\frac{1}{4}t\right)=0$

Thus, from the variation table, the range of the x-coordinate of point Q is

$$-\frac{\sqrt{3}}{36} \leqq x \leqq \frac{3}{4}$$

0196b

(2) Let α be the value of θ when point Q is on the y-axis. Then, find α and definite integral $\int_0^{\alpha}[r(\theta)]^2 d\theta$.

[Sol] From (1), $x=t^3-\frac{1}{4}t=t\left(t+\frac{1}{2}\right)\left(t-\frac{1}{2}\right)$

When $x=0$ in $0<t\leqslant 1$, $t=\frac{1}{2}$

Therefore, since $\cos\theta=\frac{1}{2}$, $\alpha=\frac{\pi}{3}$ ← Since $0\leqslant\theta<\frac{\pi}{2}$

Also, from ① and ②,

$$[r(\theta)]^2=\left[\frac{1}{2}\left(\cos\theta+\frac{1}{2}\cos 3\theta\right)\right]^2+\left[\frac{1}{2}\left(\sin\theta+\frac{1}{2}\sin 3\theta\right)\right]^2$$

$$=\frac{1}{4}\left(\cos^2\theta+\cos\theta\cos 3\theta+\frac{1}{4}\cos^2 3\theta\right)$$

$$+\frac{1}{4}\left(\sin^2\theta+\sin\theta\sin 3\theta+\frac{1}{4}\sin^2 3\theta\right)$$

$$=\frac{5}{16}+\frac{1}{4}(\cos\theta\cos 3\theta+\sin\theta\sin 3\theta)$$

$$=\frac{5}{16}+\frac{1}{4}\left[\frac{1}{2}(\cos 4\theta+\cos 2\theta)-\frac{1}{2}(\cos 4\theta-\cos 2\theta)\right]$$

$$=\frac{5}{16}+\frac{1}{4}\cos 2\theta$$

$[r(\theta)]^2=OQ^2$
$=x^2+y^2$

$\cos\theta\cos 3\theta$
$=\frac{1}{2}[\cos(\theta+3\theta)+\cos(\theta-3\theta)]$
$\sin\theta\sin 3\theta$
$=-\frac{1}{2}[\cos(\theta+3\theta)-\cos(\theta-3\theta)]$
$\cos(-2\theta)=\cos 2\theta$

Thus,

$$\int_0^{\alpha}[r(\theta)]^2 d\theta=\int_0^{\frac{\pi}{3}}\left(\frac{5}{16}+\frac{1}{4}\cos 2\theta\right)d\theta$$

$$=\left[\frac{5}{16}\theta+\frac{1}{8}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$=\frac{5}{48}\pi+\frac{\sqrt{3}}{16}$$

Applications of Calculus 2

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For real numbers a and b, line $l : y=ax+b$ is tangent to curve $C : y=\ln(x+1)$ at the point where the x-coordinate satisfies $0 \leqslant x \leqslant e-1$.

(1) Find the range for which point (a, b) exists and draw it on the ab-plane.

[Sol] Since $y=\ln(x+1)$, $y'=\dfrac{1}{x+1}$

Let the coordinates of the tangent point be $(t, \ln(t+1))$. The equation of the tangent is $y-\ln(t+1)=\dfrac{1}{t+1}(x-t)$.

Since this line overlaps with line l,

When two lines $y=m_1x+n_1$ and $y=m_2x+n_2$ overlap, $m_1=m_2$ and $n_1=n_2$.

$$a=\frac{1}{t+1},\ b=-\frac{t}{t+1}+\ln(t+1),\ 0\leqslant t\leqslant e-1$$

$\ln N=-\ln\dfrac{1}{N}$ Substituting $\dfrac{1}{t+1}=a$ to express b in terms of a

Then, $b=\dfrac{1}{t+1}-1-\ln\dfrac{1}{t+1}=a-\ln a-1$

Also, since $0\leqslant t\leqslant e-1$, $\dfrac{1}{e}\leqslant a\leqslant 1$ ← Since $1\leqslant t+1\leqslant e$, $1\geqslant\dfrac{1}{t+1}\geqslant\dfrac{1}{e}$

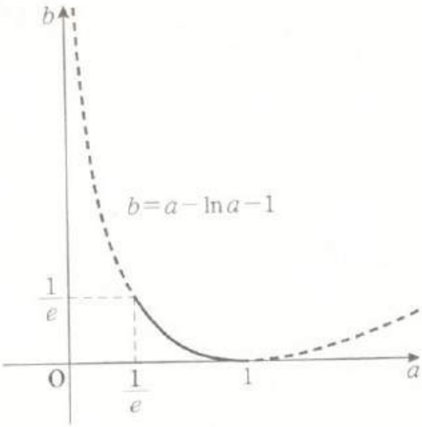
Therefore, the range for which point (a, b) exists is $\dfrac{1}{e}\leqslant a\leqslant 1$ **within** $b=a-\ln a-1$.

$$b'=1-\frac{1}{a}=\frac{a-1}{a},\ b''=\frac{1}{a^2}$$

In $\dfrac{1}{e}<a<1$, there is no a that satisfies $b'=0,\ b''=0$.

Creating the variation table,

a	$\frac{1}{e}$	$\cdots$	1
b'		$-$	
b''		$+$	
b	$\frac{1}{e}$	↘	0

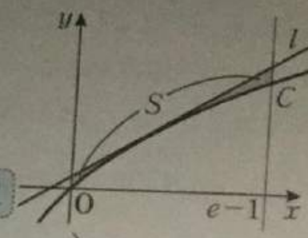


Thus, the range for which point (a, b) exists is shown as the solid line in the diagram on the right.

O197b

(2) Find the values of a and b which minimize the area enclosed by curve C and three lines l, $x=0$ and $x=e-1$. Then, find its area. ➡ O36, O132

[Sol] Let S be the area enclosed by curve C and three lines l, $x=0$ and $x=e-1$.



Since $ax+b \geqq \ln(x+1)$ in $0 \leqq x \leqq e-1$,

$$S=\int_0^{e-1}[(ax+b)-\ln(x+1)]dx$$

※1

$$=\left[\frac{a}{2}x^2+bx\right]_0^{e-1}-\left\{\left[x\ln(x+1)\right]_0^{e-1}-\int_0^{e-1}\frac{x}{x+1}dx\right\}$$

$$=\frac{a}{2}(e-1)^2+b(e-1)-(e-1)+\int_0^{e-1}\left(1-\frac{1}{x+1}\right)dx$$

$$=\frac{a}{2}(e-1)^2+b(e-1)-(e-1)+\left[x-\ln(x+1)\right]_0^{e-1}$$

$$=\frac{a}{2}(e-1)^2+b(e-1)-1$$

※1 $\int \ln(x+1)dx=\int (x)'\ln(x+1)dx$
$=x\ln(x+1)-\int x[\ln(x+1)]'dx$
$=x\ln(x+1)-\int x\cdot\frac{1}{x+1}dx$

From (1), since $b=a-\ln a-1$,

$$S=\frac{a}{2}(e-1)^2+(a-\ln a-1)(e-1)-1=\frac{e^2-1}{2}a-(e-1)\ln a-e$$

$$\frac{dS}{da}=\frac{e^2-1}{2}-\frac{e-1}{a}=\frac{(e-1)[(e+1)a-2]}{2a}$$

When $\frac{dS}{da}=0$ in $\frac{1}{e}<a<1$, $a=\frac{2}{e+1}$

a	$\frac{1}{e}$	$\cdots$	$\frac{2}{e+1}$	$\cdots$	1
$\frac{dS}{da}$		$-$	0	$+$	
S		↘	Relative minimum	↗	

From the variation table, S is a minimum at $a=\frac{2}{e+1}$.

※2

Then, $b=\frac{2}{e+1}-\ln\frac{2}{e+1}-1=\ln\frac{e+1}{2}-\frac{e-1}{e+1}$

$$S=\frac{e^2-1}{2}\cdot\frac{2}{e+1}-(e-1)\ln\frac{2}{e+1}-e=(e-1)\ln\frac{e+1}{2}-1$$

Therefore, **S has a minimum value $(e-1)\ln\frac{e+1}{2}-1$, at $a=\frac{2}{e+1}$ and $b=\ln\frac{e+1}{2}-\frac{e-1}{e+1}$.**

$\left[b=-\ln\frac{2}{e+1}-\frac{e-1}{e+1},\ S=-(e-1)\ln\frac{2}{e+1}-1\right]$

※2 From the variation table, it is possible to see that S has a minimum value at $a=\frac{2}{e+1}$. Therefore, it is not necessary to write S at $a=\frac{1}{e}$ and $a=1$.

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Applications of Calculus 2

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Given that n is a natural number greater than or equal to 2, let $S_n=\sum_{k=n}^{n^3-1}\frac{1}{k\ln k}$.

Solve the following questions.

(1) Find $\int_n^{n^3}\frac{dx}{x\ln x}$.

➡ O103

[Sol] Let $\ln x=t$. $\frac{1}{x}dx=dt$

x	$n \longrightarrow n^3$
t	$\ln n \longrightarrow 3\ln n$

$$\therefore \int_n^{n^3}\frac{dx}{x\ln x}=\int_{\ln n}^{3\ln n}\frac{1}{t}dt$$
$$=\Big[\ln|t|\Big]_{\ln n}^{3\ln n}$$
$$=\ln|3\ln n|-\ln|\ln n|$$
$$=\ln\left|\frac{3\ln n}{\ln n}\right|$$
$$=\ln 3$$

$\ln M-\ln N=\ln\frac{M}{N}$

(2) Given that k is a natural number greater than or equal to 2, prove the inequality $\frac{1}{(k+1)\ln(k+1)}<\int_k^{k+1}\frac{dx}{x\ln x}<\frac{1}{k\ln k}$.

➡ O124

[Sol] Let $f(x)=\frac{1}{x\ln x}$ in $x\geqslant 2$.

$$f'(x)=-\frac{1\cdot\ln x+x\cdot\frac{1}{x}}{(x\ln x)^2}=-\frac{\ln x+1}{(x\ln x)^2}<0$$

When $x\geqslant 2$, $\ln x>0$

Therefore, $f(x)$ decreases in $x\geqslant 2$.

Thus, when $k\geqslant 2$, $f(k+1)\leqslant f(x)\leqslant f(k)$ in $k\leqslant x\leqslant k+1$.

$$\therefore \frac{1}{(k+1)\ln(k+1)}\leqslant\frac{1}{x\ln x}\leqslant\frac{1}{k\ln k}$$

Also, the equality sign does not hold for all values of x.

$$\therefore \int_k^{k+1}\frac{dx}{(k+1)\ln(k+1)}<\int_k^{k+1}\frac{dx}{x\ln x}<\int_k^{k+1}\frac{dx}{k\ln k}$$

$$\therefore \frac{1}{(k+1)\ln(k+1)}<\int_k^{k+1}\frac{dx}{x\ln x}<\frac{1}{k\ln k}$$

$\int_k^{k+1}\frac{dx}{(k+1)\ln(k+1)}=\frac{1}{(k+1)\ln(k+1)}\Big[x\Big]_k^{k+1}=\frac{1}{(k+1)\ln(k+1)}$, $\int_k^{k+1}\frac{dx}{k\ln k}=\frac{1}{k\ln k}\Big[x\Big]_k^{k+1}=\frac{1}{k\ln k}$

O198b

(3) Find the value of $\lim_{n\to\infty} S_n$.

[Sol] $\dfrac{1}{(k+1)\ln(k+1)} < \displaystyle\int_k^{k+1} \frac{dx}{x\ln x} < \frac{1}{k\ln k}$ $\cdots$①

Substituting $k=n,\ n+1,\ \ldots,\ n^3-1$ into ① and adding up the terms on each side, when $n \geqq 2$,

O129b

$$\sum_{k=n}^{n^3-1} \frac{1}{(k+1)\ln(k+1)} < \sum_{k=n}^{n^3-1} \int_k^{k+1} \frac{dx}{x\ln x} < \sum_{k=n}^{n^3-1} \frac{1}{k\ln k} \cdots ②$$

$$\sum_{k=n}^{n^3-1} \frac{1}{(k+1)\ln(k+1)} = \frac{1}{(n+1)\ln(n+1)} + \frac{1}{(n+2)\ln(n+2)} + \cdots + \frac{1}{(n^3-1)\ln(n^3-1)} + \frac{1}{n^3\ln n^3}$$

$$= S_n - \frac{1}{n\ln n} + \frac{1}{n^3\ln n^3} \cdots ③$$

※1

$$\sum_{k=n}^{n^3-1} \int_k^{k+1} \frac{dx}{x\ln x} = \int_n^{n+1} \frac{dx}{x\ln x} + \int_{n+1}^{n+2} \frac{dx}{x\ln x} + \cdots + \int_{n^3-1}^{n^3} \frac{dx}{x\ln x}$$

$$= \int_n^{n^3} \frac{dx}{x\ln x}$$

From (1)

$$= \ln 3 \cdots ④$$

$$\sum_{k=n}^{n^3-1} \frac{1}{k\ln k} = S_n \cdots ⑤$$

From ②~⑤, $S_n - \dfrac{1}{n\ln n} + \dfrac{1}{n^3\ln n^3} < \ln 3 < S_n$

$\therefore\ \ln 3 < S_n < \ln 3 + \dfrac{1}{n\ln n} - \dfrac{1}{n^3\ln n^3}$

※2

Then, since $\lim_{n\to\infty}\left(\ln 3 + \dfrac{1}{n\ln n} - \dfrac{1}{n^3\ln n^3}\right) = \ln 3$,

※3

$\lim_{n\to\infty} S_n = \mathbf{\ln 3}$

※1 $\dfrac{1}{(n+1)\ln(n+1)} + \dfrac{1}{(n+2)\ln(n+2)} + \cdots + \dfrac{1}{(n^3-1)\ln(n^3-1)} + \dfrac{1}{n^3\ln n^3}$
$= \displaystyle\sum_{k=n}^{n^3-1} \frac{1}{k\ln k} - \frac{1}{n\ln n} + \frac{1}{n^3\ln n^3}$

※2 Since $S_n - \dfrac{1}{n\ln n} + \dfrac{1}{n^3\ln n^3} < \ln 3$, $S_n < \ln 3 + \dfrac{1}{n\ln n} - \dfrac{1}{n^3\ln n^3}$

Combined with $\ln 3 < S_n$, $\ln 3 < S_n < \ln 3 + \dfrac{1}{n\ln n} - \dfrac{1}{n^3\ln n^3}$

※3 Limits of Sequences and Their Relationships (N67)

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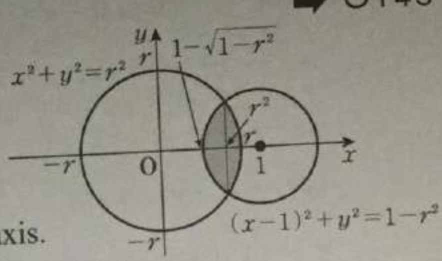
Let $0<r<1$. In a space, let $V(r)$ be the volume of the overlapping area of the sphere with its center at point $(0, 0, 0)$ and radius r and the sphere with its center at point $(1, 0, 0)$ and radius $\sqrt{1-r^2}$. Solve the following questions.

➡ O145

(1) Find $V(r)$.



[Sol] $V(r)$ is the volume of the solid formed by rotating the overlapping area of two circles $x^2+y^2=r^2$ and $(x-1)^2+y^2=1-r^2$ once about the x-axis.

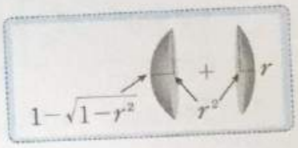
The x-coordinate of the points of intersection of the two circles is, since $(x-1)^2+(r^2-x^2)=1-r^2$, $x=r^2$

Substituting $y^2=r^2-x^2$ into $(x-1)^2+y^2=1-r^2$

Also, the x-coordinates of the points of intersection of $(x-1)^2+y^2=1-r^2$ and the x-axis are, since $(x-1)^2=1-r^2$, $x=1\pm\sqrt{1-r^2}$

Substituting $y=0$ into $(x-1)^2+y^2=1-r^2$

$$\therefore\ V(r)=\pi\int_{1-\sqrt{1-r^2}}^{r^2}[1-r^2-(x-1)^2]dx+\pi\int_{r^2}^{r}(r^2-x^2)dx$$



Therefore,

$$\int_{1-\sqrt{1-r^2}}^{r^2}[1-r^2-(x-1)^2]dx$$
$$=\left[(1-r^2)x-\frac{1}{3}(x-1)^3\right]_{1-\sqrt{1-r^2}}^{r^2}$$
$$=\left[(1-r^2)r^2-\frac{1}{3}(r^2-1)^3\right]-\left[(1-r^2)(1-\sqrt{1-r^2})-\frac{1}{3}(-\sqrt{1-r^2})^3\right]$$
$$=\frac{2}{3}(1-r^2)\sqrt{1-r^2}-\frac{1}{3}r^6+r^2-\frac{2}{3}$$

$$\int_{r^2}^{r}(r^2-x^2)dx=\left[r^2x-\frac{1}{3}x^3\right]_{r^2}^{r}=\frac{1}{3}r^6-r^4+\frac{2}{3}r^3$$

Thus,

$$V(r)=\pi\left[\frac{2}{3}(1-r^2)\sqrt{1-r^2}-\frac{1}{3}r^6+r^2-\frac{2}{3}\right]+\pi\left(\frac{1}{3}r^6-r^4+\frac{2}{3}r^3\right)$$
$$=\frac{\pi}{3}[2(1-r^2)\sqrt{1-r^2}-3r^4+2r^3+3r^2-2]$$

O199b

(2) In the interval $0 < r < 1$, find the value of r which maximizes $V(r)$ and the maximum value of $V(r)$.

[Sol] From (1),

$$V'(r)=\frac{\pi}{3}\left[2\cdot\frac{3}{2}\sqrt{1-r^2}\cdot(-2r)-12r^3+6r^2+6r\right]$$
$$=2\pi r(-\sqrt{1-r^2}-2r^2+r+1)$$

When $V'(r)=0$ in $0 < r < 1$,

$$-\sqrt{1-r^2}-2r^2+r+1=0$$

$$\sqrt{1-r^2}=-2r^2+r+1$$

Squaring both sides and simplifying

$$2r^4-2r^3-r^2+r=0$$

The Factor Theorem (J172)

$$r(r-1)(2r^2-1)=0$$

Since $0 < r < 1$, $r=\frac{\sqrt{2}}{2}$

r	0	$\cdots$	$\frac{\sqrt{2}}{2}$	$\cdots$	1
$V'(r)$		$+$	0	$-$	
$V(r)$		$\nearrow$	$\frac{\pi}{3}\left(\sqrt{2}-\frac{5}{4}\right)$	$\searrow$	

From the variation table,

the maximum value of $V(r)$ is $\frac{\pi}{3}\left(\sqrt{2}-\frac{5}{4}\right)$, at $r=\frac{\sqrt{2}}{2}$.

$$\left[=\frac{(4\sqrt{2}-5)\pi}{12}\right]$$

Applications of Calculus 2

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The function $f(x)=xe^{-x^3}$ is defined for all real numbers.

(1) For the given $f(x)$, determine where the function increases/decreases and its concavity. Then, sketch its graph. ➡ O25

[Sol] $f'(x)=1\cdot e^{-x^3}+xe^{-x^3}\cdot(-3x^2)=(1-3x^3)e^{-x^3}$

$f''(x)=-9x^2e^{-x^3}+(1-3x^3)e^{-x^3}\cdot(-3x^2)=-3x^2(4-3x^3)e^{-x^3}$

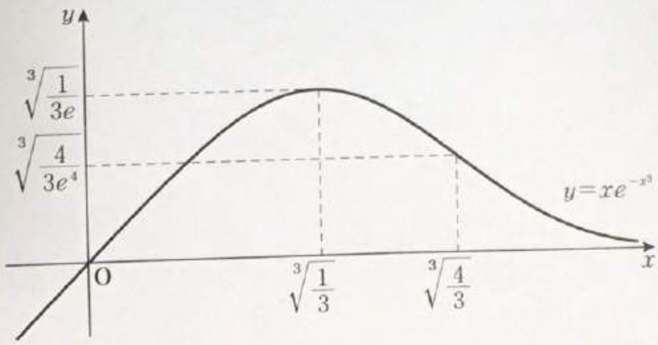
When $f'(x)=0$, $x=\sqrt[3]{\frac{1}{3}}$. When $f''(x)=0$, $x=0$, $\sqrt[3]{\frac{4}{3}}$

Creating the variation table,

x	$\cdots$	0	$\cdots$	$\sqrt[3]{\frac{1}{3}}$	$\cdots$	$\sqrt[3]{\frac{4}{3}}$	$\cdots$
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$-$	0	$-$	$-$	$-$	0	$+$
$f(x)$	↗	0	↗	$\sqrt[3]{\frac{1}{3e}}$	↘	$\sqrt[3]{\frac{4}{3e^4}}$	↘

Also, $\lim_{x\to\infty} f(x)=0$

From the above, the graph is as shown below.



$\lim_{x\to\infty} xe^{-x^3}=0$

O200b

(2) For a positive number C, let D_1 be the area enclosed by $y=f(x)$, the x-axis and $x=C$. Let $V_1(C)$ be the volume of the solid formed by rotating D_1 once about the x-axis. Find $\lim_{C\to\infty} V_1(C)$.

➡ O143

[Sol] $V_1(C)=\pi\int_0^C (xe^{-x^3})^2dx$

$$=\pi\int_0^C x^2e^{-2x^3}dx$$

$$=\pi\left[-\frac{1}{6}e^{-2x^3}\right]_0^C$$

$$=\frac{\pi}{6}(1-e^{-2C^3})$$

Since $\left(-\frac{1}{6}e^{-2x^3}\right)'=-\frac{1}{6}e^{-2x^3}\cdot(-6x^2)=x^2e^{-2x^3}$,
$\int x^2e^{-2x^3}dx=-\frac{1}{6}e^{-2x^3}+A$
(A is the constant of integration)

Since $\lim_{C\to\infty} e^{-2C^3}=0$, $\lim_{C\to\infty} V_1(C)=\frac{\pi}{6}$

(3) Let M be the maximum value of $y=f(x)$ in $x\geqq 0$, and let D_2 be the area enclosed by $y=f(x)$, the y-axis and $y=M$. Given the solid formed by rotating D_2 once about the y-axis, find the volume V_2 of this solid.

➡ O144

[Sol] From (1), since $M=\sqrt[3]{\frac{1}{3e}}$, $V_2=\pi\int_0^{\sqrt[3]{\frac{1}{3e}}} x^2dy$

Since $y=f(x)$, $dy=f'(x)dx$ ※

y	$0 \longrightarrow \sqrt[3]{\frac{1}{3e}}$
x	$0 \longrightarrow \sqrt[3]{\frac{1}{3}}$

Therefore,

$$V_2=\pi\int_0^{\sqrt[3]{\frac{1}{3}}} x^2f'(x)dx$$

$$=\pi\left\{\left[x^2f(x)\right]_0^{\sqrt[3]{\frac{1}{3}}}-2\int_0^{\sqrt[3]{\frac{1}{3}}} xf(x)dx\right\}$$

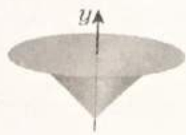
$\int x^2f'(x)dx$
$=x^2f(x)-\int(x^2)'f(x)dx$

$$=\frac{\pi}{3}\sqrt[3]{\frac{1}{e}}-2\pi\int_0^{\sqrt[3]{\frac{1}{3}}} x^2e^{-x^3}dx$$

$$=\frac{\pi}{3}\sqrt[3]{\frac{1}{e}}-2\pi\left[-\frac{1}{3}e^{-x^3}\right]_0^{\sqrt[3]{\frac{1}{3}}}$$

Since $\left(-\frac{1}{3}e^{-x^3}\right)'=-\frac{1}{3}e^{-x^3}\cdot(-3x^2)=x^2e^{-x^3}$,
$\int x^2e^{-x^3}dx=-\frac{1}{3}e^{-x^3}+A$
(A is the constant of integration)

$$=\pi\left(\sqrt[3]{\frac{1}{e}}-\frac{2}{3}\right)$$



※ Since it is difficult to express x as the function of y, try to replace dy with dx.